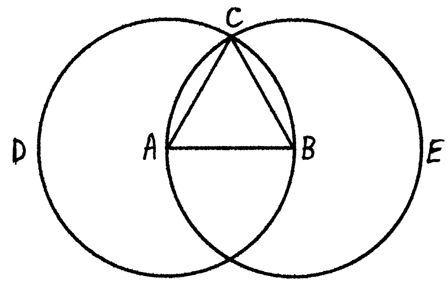
Selected Proofs[1] from ***The Elements, Book I***

Theorem 1 *Construction of an equilateral triangle, given one side.*

Proof:

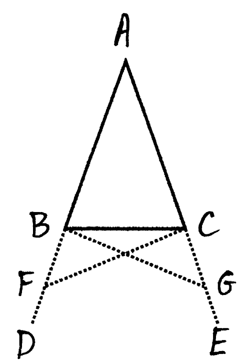
1. Given line AB.
2. With A as center and using AB as the radius draw circle BCD (see drawing). With B as center and using AB as the radius draw circle ACE. (Post. 3)
3. From point C, where the two circles intersect, draw lines to both points A and B. (Post. 1).
4. Since point A is the center of circle BCD, AC ≅ AB
   Since point B is the center of circle ACE, BC ≅ AB
   (Def. of a circle)
5. AC ≅ BC (C.N. 1)
6. AC ≅ AB ≅ BC ∴ΔABC is equilateral. (Definition of equilateral) Q.E.D.



Theorem 5 *In an Isosceles triangle, (a) the base angles are equal to one another, and (b) if the two sides are extended, then the angles under the bases will be equal to one another.*

Proof:

1. Given ΔABC is an Isosceles triangle. Let AB and AC be the equal sides. (Def. of Isosceles triangle)
2. Extend sides AB and AC to D and E, respectively. (Post. 2)
3. Choose point F at random on BD. Cut off AE at G, such that AG ≅ AF. (Th. I-3)
4. Join the lines FC and GB. (Post. 1)
5. Regarding ΔAFC and ΔAGB :
   they share ∠GAF; AF ≅ AG (step 3); AB ≅ AC (step 1)
   ∴ΔAFC ≅ ΔAGB; FC ≅ BG; ∠ACF ≅ ∠ABG;
   and ∠AFC ≅ ∠AGB (Th. I-4)
6. Since AF ≅ AG and AB ≅ AC then BF ≅ CG (C.N. 3)
7. Regarding ΔBFC and ΔCGB :
   ∠BFC ≅ ∠CGB (same as ∠AFC ≅ ∠AGB, step 5)
   BF ≅ CG (step 7) and FC ≅ BG (step 5)
   ∴ ΔBFC ≅ ΔCGB, ∠BCF ≅ ∠CBG, and
   ∠CBF ≅ ∠BCG (Th. I-4) Q.E.D. (for part b)
8. Since ∠ACF ≅ ∠ABG (step 5) and in these angles ∠BCF ≅ ∠CBG (step 7)
   Then the remaining angles are equal (C.N. 3):
   ∴∠ABC ≅ ∠ACB Q.E.D. (for part a)

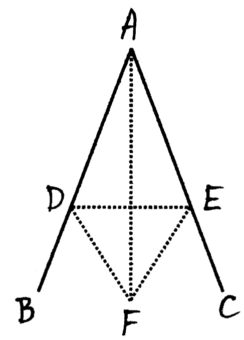


[1] All of the proofs listed here are the result of me re-wording T. L. Heath's translation of *The Elements* (Dover Publications, 1956).

Selected Proofs from *The Elements, Book I* (continued)

Theorem 9 *Construction of an angle bisector.*

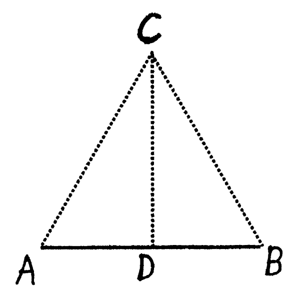
Proof:



1. Given $\angle BAC$ (to be bisected) with point D randomly on AB.
2. Let AC be cut off at E, such that $AD \cong AE$ (Th. I-3)
3. Draw DE. (Post. 1)
4. Draw equilateral ΔDEF on DE. (Th. I-1)
5. Draw AF. (Post. 1)
6. $DF \cong EF$ (Def. of Equilateral Triangle)
7. $\angle DAF \cong \angle EAF$ (Th. I-8 SSS)
8. $\therefore \angle BAC$ has been bisected. (Def. of Bisect) Q.E.D.

Theorem 10 *Bisection of a line.*

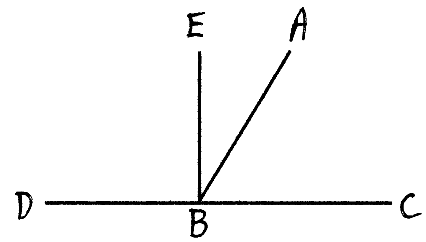
Proof:



1. Given line AB to be bisected.
2. Draw equilateral ΔABC on AB (Th. I-1)
3. Draw the bisector CD of $\angle ACB$ (Th. I-9)
4. $\angle ACD \cong \angle BCD$ (Def. Angle Bisector)
5. $AC \cong BC$ (Def. of Equilateral Triangle)
6. $\Delta ACD \cong \Delta BCD$ and $AD \cong BD$ (Th. I-4)
7. $\therefore$ AB has been bisected. (Def. of Bisect) Q.E.D.

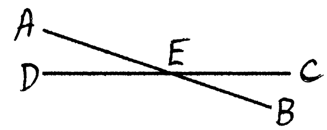
Theorem 13 *Supplementary Angle Theorem* (Y Theorem). *If two adjacent angles form a straight line, then the sum of the angles is equal to two right angles.*

Proof:



1. Given line AB set up on line DC.
2. If $\angle CBA \cong \angle ABD$ then they are two right angles. (Definition of Right Angles)
3. If these two angles are not equal, then draw BE from point B and perpendicular to line DC. (Th. I-11)
4. $\angle CBE$ and $\angle DBE$ are right angles. (Def. of Perpendicular)
5. $m\angle CBE = m\angle ABC + m\angle ABE$ (from drawing: No reason stated)
6. $m\angle CBE + m\angle DBE = m\angle DBE + m\angle ABC + m\angle ABE$ (C.N. 2)
7. $m\angle DBA = m\angle DBE + m\angle ABE$ (from drawing: No reason stated)
8. $m\angle DBA + m\angle ABC = m\angle DBE + m\angle ABE + m\angle ABC$ (C.N. 2)
9. $m\angle CBE + m\angle DBE = m\angle DBA + m\angle ABC$ (steps 6 & 8, C.N. 1)
10. Because $\angle CBE$ and $\angle DBE$ are both right angles (step 4), $\angle DBA$ and $\angle ABC$ together form two right angles [they are supplementary]. (C.N. 1) Q.E.D.

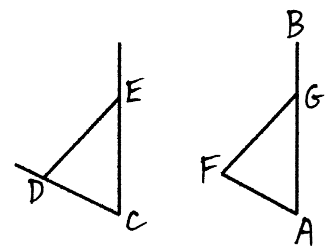
Selected Proofs from *The Elements, Book I* (continued)



Theorem 15 (X Theorem) *Vertical angles are equal.*

Proof:

1. Given lines AB and CD intersecting at E.
2. The sum of ∠CEA and ∠AED is equal to two right angles. (Th. I-13)
3. The sum of ∠CEA and ∠CEB is equal to two right angles. (Th. I-13)
4. The sum of ∠CEA and ∠AED is equal to the sum of ∠CEA and ∠CEB (Post. 4 and C.N. 1)
5. ∠AED ≅ ∠CEB (C.N. 3) (Similarly, it can be proven that ∠CEA ≅ ∠BED) Q.E.D.



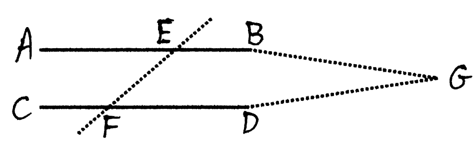
Theorem 23 *Copying an angle.*

Proof:

1. Given angle DCE to be copied to point A on the line AB.
2. Let DE be drawn. (Post. 1)
3. By using the three lines CD, DE, and CE construct the triangle AFG such that CD = AF, CE = AG, and DE = FG. (Th. I-22)
4. Since the three sides of the triangle AFG are equal to the three sides of the triangle CDE, then the angle DCE is equal to the angle FAG. (Th. I-8) *Q.E.D.*

Theorem 27 *If two lines are cut by a transversal, and alternate interior angles are equal, then the lines are parallel.*

Proof:

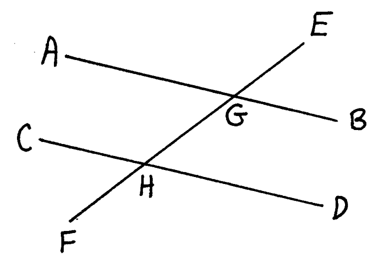


1. Given that EF falls on AE and CD, and ∠AEF ≅ ∠EFD
2. Assume that AB and CD meet at point G, in the direction of B, D.
3. Then in ΔEFG, the exterior angle (∠AEF) is equal to an interior and opposite angle (∠EFG), which is impossible. (Th. I-16)
4. ∴ the assumption (step 2) is false. AB and CD cannot meet in the direction of B, D.
5. Similarly, it can be shown that AB and CD cannot meet in the direction of A, C.
6. AB and CD do not meet in either direction, therefore they are parallel. (Def. of parallel) Q.E.D.

Selected Proofs from *The Elements, Book I* (continued)

Theorem 29 *If two parallel lines are cut by a transversal, then the alternate interior angles are equal, the corresponding angles are equal,* and *the same-side interior angles add to two right angles.*

[Note: This is the first theorem where Euclid uses the fifth postulate.]

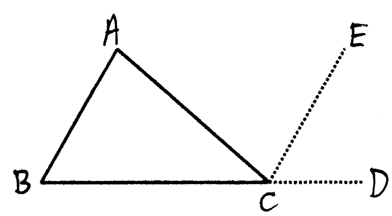


Proof:

1. Given parallel lines AB and CD, with line EF falling on them.
2. Assume that $\angle AGH$ and $\angle GHD$ are not equal, and that $\angle AGH$ is larger. So, $\angle AGH > \angle GHD$.
3. $\angle AGH + \angle BGH > \angle GHD + \angle BGH$ (C.N.2 for inequalities)
4. $\angle AGH + \angle BGH =$ two right angles (Th. I-13)
5. two right angles $> \angle GHD + \angle BGH$ (C.N.1 for inequalities)
6. Because the sum of $\angle GHD$ and $\angle BGH$ is less than two right angles, lines AB and CD must meet. (Postulate 5)
7. But lines AB and CD cannot meet. (Def. of Parallel Lines)
8. Steps 6 and 7 are in contradiction, so our assumption (step 2) must be false, and $\therefore$ $\angle AGH \cong \angle GHD$ [*alternate interior angles are equal*].
9. $\angle AGH \cong \angle EGB$ (Th. I-15)
10. $\therefore$ $\angle EGB \cong \angle GHD$ (C.N.1) [*corresponding angles are equal*]
11. $\angle EGB + \angle BGH = \angle GHD + \angle BGH$ (C.N.2)
12. $\angle EGB + \angle BGH =$ two right angles (Th. I-13)
13. $\therefore$ $\angle GHD + \angle BGH =$ two right angles (C.N.1) [*same-side interior angles add to two right angles*] Q.E.D.

Theorem 32 *In any triangle, (a) Any exterior angle is equal to the sum of the two opposite interior angles, and (b) The three interior angles add to two right angles.*

Proof:

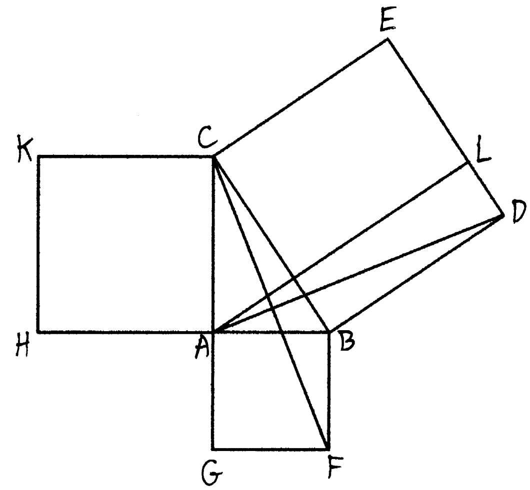


1. Given ΔABC
2. Extend BC to D (Post. 2)
3. Draw CE parallel to AB (Th. I-31)
4. Since AB is parallel to CE, and AC transverses both of them, $\angle ACE \cong \angle BAC$ (Th. I-29) and $\angle ECD \cong \angle ABC$ (Th. I-29)
5. $\angle ACE + \angle ECD = \angle BAC + \angle ABC$ (C.N. 2)
6. $\angle ACE + \angle ECD = \angle ACD$ (from drawing)
7. **$\angle ACD = \angle BAC + \angle ABC$** (C.N. 1) Q.E.D. (for part a)
8. Adding $\angle ACB$ to both sides of equation: $\angle ACD + \angle ACB = \angle BAC + \angle ABC + \angle ACB$ (C.N. 2)
9. But $\angle ACD$ and $\angle ACB$ are adjacent angles and form the straight line BD, therefore they are equal to two right angles. (Th. I-13)
10. $\therefore$ **$\angle BAC + \angle ABC + \angle ACB$ is also equal to two right angles.** (C.N. 1) Q.E.D. (for part b)

Selected Proofs from *The Elements, Book I* (continued)

Euclid's Proof of the Pythagorean Theorem

(Theorem I-47)



1. Given: ΔABC is a right triangle, with $\angle$BAC a right angle.
2. Construct a square on each of the 3 sides of ΔABC. (Th. I-46)
3. Draw AL parallel to BD. (Th. I-31)
4. Draw lines AD and FC. (Post. 1)
5. (a) $\angle$DBC & $\angle$FBA are both right angles. (Def. of Square)
   (b) $\angle$DBC $\cong$ $\angle$FBA (Post. 4)
   (c) $\angle$DBC + $\angle$ABC = $\angle$FBA + $\angle$ABC (C.N. 2)
   (d) $\angle$ABD $\cong$ $\angle$FBC (from drawing)
6. (a) BD $\cong$ BC and AB $\cong$ FB. (Def. of Square)
   (b) ΔABD $\cong$ ΔFBC because BD $\cong$ BC and AB $\cong$ FB and $\angle$ABD $\cong$ $\angle$FBC (step 6). (Th. I-4)
7. (a) $\angle$BAG is a right angle. (Def. of Square)
   (b) $\angle$BAC and $\angle$BAG are adjacent and both right angles, so CA is in a straight line with AG. (Th. I-14)
   (c) $\angle$BAC $\cong$ $\angle$FBA (Post. 4)
   (d) CG is parallel to FB. (Th. I-27)
   (e) [The area of] square GB is twice [the area of] ΔFBC, because they have the same base FB and lie between the same parallels FB and GC. (Th. I-41)
8. [The area of] parallelogram BL is twice [the area of] ΔABD, because they have the same base BD and they lie between the same parallels BD and AL. (Th. I-41)
9. ΔFBC $\cong$ ΔABD, therefore twice [the area of] ΔFBC is equal to twice [the area of] ΔABD. (C.N. 1)
10. [The area of] square GB is equal to [the area of] parallelogram BL. (C.N. 1)
11. Similarly, if lines AE and BK are drawn, parallelogram CL can be proven equal to square HC.
12. The sum of [the areas of] squares HC and GB is equal to
    the sum of [the areas of] parallelograms CL and BL. (C.N. 2)
13. [The area of] the square BE is equal to the sum of [the areas of] parallelograms CL and BL. (fr. drawing)
14. $\therefore$ **[The area of] the square BE is equal to the sum of [the areas of] the squares GB and HC.**
    (C.N. 1) *Q.E.D.*

Theorem X-1 from *The Elements*[2]

In the Original Greek

Δύο μεγεθῶν ἀνίσων ἐκκειμένων, ἐὰν ἀπὸ τοῦ μείζονος ἀφαιρεθῇ μεῖζον ἢ τὸ ἥμισυ καὶ τοῦ καταλειπομένου μεῖζον ἢ τὸ ἥμισυ, καὶ τοῦτο ἀεὶ γίγνηται, λειφθήσεταί τι μέγεθος, ὃ ἔσται ἔλασσον τοῦ ἐκκειμένου ἐλάσσονος μεγέθους.

Ἔστω δύο μεγέθη ἄνισα τὰ ΑΒ, Γ, ὧν μεῖζον τὸ ΑΒ· λέγω, ὅτι ἐὰν ἀπὸ τοῦ ΑΒ ἀφαιρεθῇ μεῖζον ἢ τὸ ἥμισυ καὶ τοῦ καταλειπομένου μεῖζον ἢ τὸ ἥμισυ, καὶ τοῦτο ἀεὶ γίγνηται, λειφθήσεταί τι μέγεθος, ὃ ἔσται ἔλασσον τοῦ Γ μεγέθους.

Τὸ Γ γὰρ πολλαπλασιαζόμενον ἔσται ποτὲ τοῦ ΑΒ μεῖζον. πεπολλαπλασιάσθω, καὶ ἔστω τὸ ΔΕ τοῦ μὲν Γ πολλαπλάσιον, τοῦ δὲ ΑΒ μεῖζον, καὶ διῃρήσθω τὸ ΔΕ εἰς τὰ τῷ Γ ἴσα τὰ ΔΖ, ΖΗ, ΗΕ, καὶ ἀφῃρήσθω ἀπὸ μὲν τοῦ ΑΒ μεῖζον ἢ τὸ ἥμισυ τὸ ΒΘ, ἀπὸ δὲ τοῦ ΑΘ μεῖζον ἢ τὸ ἥμισυ τὸ ΘΚ, καὶ τοῦτο ἀεὶ γιγνέσθω, ἕως ἂν αἱ ἐν τῷ ΑΒ διαιρέσεις ἰσοπληθεῖς γένωνται ταῖς ἐν τῷ ΔΕ διαιρέσεσιν.

Ἔστωσαν οὖν αἱ ΑΚ, ΚΘ, ΘΒ διαιρέσεις ἰσοπληθεῖς οὖσαι ταῖς ΔΖ, ΖΗ, ΗΕ· καὶ ἐπεὶ μεῖζόν ἐστι τὸ ΔΕ τοῦ ΑΒ, καὶ ἀφῄρηται ἀπὸ μὲν τοῦ ΔΕ ἔλασσον τοῦ ἡμίσεως τὸ ΕΗ, ἀπὸ δὲ τοῦ ΑΒ μεῖζον ἢ τὸ ἥμισυ τὸ ΒΘ, λοιπὸν ἄρα τὸ ΗΔ λοιποῦ τοῦ ΘΑ μεῖζόν ἐστιν. καὶ ἐπεὶ μεῖζόν ἐστι τὸ ΗΔ τοῦ ΘΑ, καὶ ἀφῄρηται τοῦ μὲν ΗΔ ἥμισυ τὸ ΗΖ, τοῦ δὲ ΘΑ μεῖζον ἢ τὸ ἥμισυ τὸ ΘΚ, λοιπὸν ἄρα τὸ ΔΖ λοιποῦ τοῦ ΑΚ μεῖζόν ἐστιν. ἴσον δὲ τὸ ΔΖ τῷ Γ· καὶ τὸ Γ ἄρα τοῦ ΑΚ μεῖζόν ἐστιν. ἔλασσον ἄρα τὸ ΑΚ τοῦ Γ.

Καταλείπεται ἄρα ἀπὸ τοῦ ΑΒ μεγέθους τὸ ΑΚ μέγεθος ἔλασσον ὂν τοῦ ἐκκειμένου ἐλάσσονος μεγέθους τοῦ Γ· ὅπερ ἔδει δεῖξαι—ὁμοίως δὲ δειχθήσεται, κἂν ἡμίση ᾖ τὰ ἀφαιρούμενα.

Translated into English

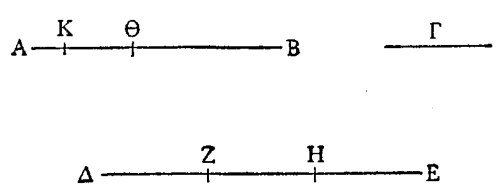
Two unequal magnitudes being set out, if from the greater there be subtracted a magnitude greater than the half, and from the remainder a magnitude greater than its half, and so on continually, there will be left some magnitude which will be less than the lesser magnitude set out.

Let AB, Γ be the two unequal magnitudes, of which AB is the greater; I say that, if from AB there be subtracted a magnitude greater than its half, and from the remainder a magnitude greater than its half, and so on continually, there will be left some magnitude which will be less than the magnitude Γ.

For Γ, if multiplied, will at some time be greater than AB [see v. Def. 4]. Let it be multiplied, and let ΔE be a multiple of Γ, greater than AB, and let ΔE be divided into the parts ΔZ, ZH, HE equal to Γ, and from AB let there be subtracted BΘ greater than its half, and from AΘ let there be subtracted ΘK greater than its half, and so on continually, until the divisions in AB are equal in multitude to the divisions in ΔE.

Let, then, AK, KΘ, ΘB be divisions equal in multitude with ΔZ, ZH, HE; now since ΔE is greater than AB, and from ΔE there has been subtracted EH less than its half, and from AB there has been subtracted BΘ greater than its half, therefore the remainder HΔ is greater than the remainder ΘA. And since HΔ is greater than ΘA, and from HΔ there has been subtracted the half, HZ, and from ΘA there has been subtracted ΘK greater than its half, therefore the remainder ΔZ is greater than the remainder AK. Now ΔZ is equal to Γ; and therefore Γ is greater than AK. Therefore AK is less than Γ.

There is therefore left of the magnitude AB the magnitude AK which is less than the lesser magnitude set out, namely, Γ; which was to be proved—and this can be similarly proved even if the parts to be subtracted be halves.



[2] Ivor Thomas, *Greek Mathematical Works*. Harvard University Press, 1998. pp 453-5

Appendix C – Euclid's *Elements*