Lesson Plans for (9th Grade Main Lesson)
Possibility & Probability
(including Permutations and Combinations) (Last updated October 2018)

Overview

The subject of *possibility and probability* (which includes permutations and combinations) concentrates on answering questions such as: How many possible (shortest) routes are there for going between two points marked on a grid? How many ways are there for 20 students to get in line? How many different committees of three can be formed from a group of 10 people? What is the probability of flipping five coins and getting all heads?

These questions often yield surprising results. It is through careful thinking that we can overcome the task of making sense of these difficult problems; we recognize patterns and similarities with previously encountered problems; and learn to solve the problems in a systematic way.

The students work both independently and in groups to solve problems. Carefully worded explanations of a few solutions are written in their lesson books. Each student is challenged at some point during the course, and hopefully, in the process, develops confidence in their thinking.

Notes for the Teacher

- At my school, there is only room for one math main lesson block in ninth grade. Therefore, the two ninth grade math main lesson blocks (*Possibility & Probability* and *Descriptive Geometry*) are both during one 4-week main lesson block. Usually, I work it so that the first week is only Possibility & Probability, and the last week is only Descriptive Geometry. Thus, each block runs for three weeks, and there are two weeks of overlap.
- Therefore, with what is listed below, days #1-5 are full days, and days #6-13 are half days.

Day #1

- Why do we teach this block in 9th grade?
  - What do you need in order to be prepared for college?
    1. **Enthusiasm for learning.**
    2. **Study skills.** The term “study skills” is broad; it includes organization, work habits, etc. We should ask: “Have they learned how to learn, and do they have confidence in their abilities to learn new material?”
    3. **Higher level thinking.** Our goal is to have students graduate high school who can think flexibly, creatively, and independently.
    4. **Basic skills.** Yes, skills are important. However, the list of necessary skills isn’t too long.
  - There is a crisis in the world today – people aren’t able to think for themselves. What happens when people watch the TV news? They are told what to think! (Example: Richard Jewell’s “trial by media”)
  - There is a lack of independent, creative and flexible thinking.

    Quotes from my book:

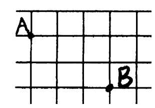
    “All too often, in today’s world, students graduate from high school and college unable to think for themselves. Their thinking, even when doing math, is largely imitative; they can do math problems as long as they have seen something similar before. As Waldorf educators, our goal is to have students graduate high school who can think flexibly, creatively, and independently. We want our students to be able to think for themselves, think analytically, and we hope that their thinking is heart-felt and imbued with imagination.”

    “Certainly, one of the ultimate goals of any educational system should be that its graduates can think independently, flexibly and creatively. Yet, we often hear from university professors and people in industry hiring for entry-level engineering positions, that today’s education isn’t able to adequately produce graduates who are good problem-solvers and can think creatively.”

- In grades 1-8, you were expected to believe what your teacher told you. E.g., your teacher may have told you that $\pi \approx 3.14159$. In high school, you need to learn to think for yourself.

  Thought for the Day: "*Don't just believe what you're told. Believe it only when you know – in your own thinking – that it's true.*"

- Course Expectations.
  - Pass out "expectations sheet" and go over it.
  - Be open minded. This is a totally new kind of math and new kind of thinking. Nobody is at a disadvantage because of weak math or algebra skills. The math involved is quite easy, but the thinking is quite challenging at times.
  - We will do a few classic, interesting, challenging problems in groups, or as a whole class.
  - You will not be expected to do very difficult problems on your own during this course. During track class later in the year, you will have a unit that is focused on solving a lot of these problems.
  - If you have learned the material fairly well, the final test should be quite manageable.
- The Street Problem. How many shortest routes are there from A to B?
  - Don't say what the answer is.



- Hanging questions:
  - With the Street Problem, how many possibilities would there be for a 5x6 grid?
  - In a group of 50 people, what is the probability that at least two have the same b-day?

Day #2

- The Wardrobe Problem. How many possible outfits can you choose to wear if you have 3 pants and 5 shirts to choose from?
- The Seating Chart Problem. How many possibilities are there for making a seating chart for the class?
  - Add a second part to the question: If there is a new seating chart every second, then how long will it take before all of the possibilities will be exhausted, but none repeated.
  - Likely, this problem won't be solved by the end of class.
- ML Book Essay: Street problem

Day #3

- The Seating Chart Problem. Go over it. Have students share how their solutions. It is also valuable, if done in the right way, to have students share solutions that don't lead to the correct answer. In the end come also to an answer of the amount of time to go through all the possible seating charts, in seconds, then hours, then days, then years.
- License Plate Problem. How many possibilities are there for a license plate that has 3 digits followed by 3 letters?
- The Prize Problem: How many possibilities are there for giving out 1^{st}, 2^{nd}, 3^{rd} place prizes in a race that has 53 participants?
- Hanging questions:
  - If you rearranged the class into a different seating chart every second, how long would it take before you exhausted every possibility?
- ML Book Essay: Wardrobe problem, and Seating chart problem.

Day #4

- Review all previous problems.
  - All of these problems are the same: How many possibilities are there to…
    - Choose an outfit if you have 10 pair of pants, 9 shirts, and 8 hats?
    - Select a "complete" meal if the menu has 8 appetizers, 10 main courses, 9 desserts.
    - Select 3 students out of 10 to sit in three seats.
    - Award 1st, 2nd, 3rd prize in a 10-person race.
    - Choose a president, VP and secretary from a club of 10 people.
- Permutation.
  - The number of possibilities for selecting, *in order,* a sub-group from a larger group. (Don't give a definition to be memorized)
  - ${}_nP_r$ means the first r numbers of n factorial multiplied together.
- The Committee Problem. First ask, how many possibilities are there for selecting a Pres, VP, and Sec from a group of 5 people. Then ask how many possibilities there are for selecting 3 players out of 5 to be on an all-star team.
- ML Book Essay: Prize problem. (License Plate problem is extra credit.)
- Hanging questions:
  - If 5 new students joined the class, how many times more possible seating charts would there be?
  - How many possible committees of 5 people can be selected from a group of 75?

Day #5

Review: Review the committee problem, then have students fill out the last two columns of following table:

Total # of people	# of people selected	# of possibilities for prize problem	# of possibilities for committee problem
5	3	${}_5P_3 = 60$	${}_5C_3 = \frac{{}_5P_3}{3!} = \frac{60}{6} = 10$
7	2	${}_7P_2 = 42$	${}_7C_2 = \frac{{}_7P_2}{2!} = \frac{42}{2} = 21$
10	3	${}_{10}P_3 = 720$	${}_{10}C_3 = \frac{{}_{10}P_3}{3!} = \frac{720}{6} = 120$
20	4	${}_{20}P_4 = 116{,}280$	${}_{20}C_4 = \frac{{}_{20}P_4}{4!} = \frac{116280}{6} = 4845$
75	5	${}_{75}P_5 = 2{,}071{,}126{,}800$	${}_{75}C_5 = \frac{{}_5P_3}{3!} = \frac{60}{6} = 17{,}259{,}390$

- ML Book Essay: Committee Problem (Double Essay) due on Tuesday (Day #7).
- Hanging questions: A preview of probability – If there are 20 blue marbles and 30 orange marbles in a bag, what is the probability of randomly pulling out one marble, and it turning out to be blue?

Day #6

The classes from here forward are only "half lessons". The other half is Descriptive Geom.

- Main Lesson Book pep talk.
  - What is the purpose of having you write these essays? Ans: It gives you a deeper level of understanding, and you get to work on your writing skills.
  - You are required to bring all of your essays tomorrow, or e-mail them to me by 9pm tonight. I want to see that you are not behind.
  - Remember, in order to get an A, your book should reflect that you have done something beyond the basic requirements. Doing only a minimal job is considered "C+" level work. Look for opportunities to do something extra every week.
  - By 11^{th} grade I don't even tell you what to put into your book.
  - No calculator for the test…but you can leave answers to problems in notation such as 5!, or ${}_7P_5$, etc.
- Possibility Summary Page (in ML Book):
  - *The Multiplication Rule* (The Wardrobe Problem). If there are X ways to choose one thing, and Y ways to choose another thing, then there are X•Y ways to choose these two things together. (This assumes that these two things are independent. For example, if I have to choose two people in the school, and one has to be a 10^{th} grader and one has to be a girl, then these two choices are *not* independent because the result of the first choice can change the number of possibilities of the second choice.)
  - *Factorials* (The Seating Chart Problem). The number of ways of rearranging n objects is n!
    For example, 5! = 5•4•3•2•1 = 120.
  - *Permutations* (The Prize Problem). The number of ways to select r out of n objects, in order, is ${}_nP_r$.
    For example, ${}_9P_4 = 9•8•7•6 = 3024$.
  - *Combinations* (The Committee Problem). The number of ways to select r out of n objects, without regard to order, is ${}_nC_r$. For example, ${}_9C_4 = \frac{{}_9P_4}{4!} = \frac{9\cdot8\cdot7\cdot6}{4\cdot3\cdot2\cdot1} = 126$.
  - *Distinguishable Arrangements* (Word Scrambling). If there are a total of N objects, of which A objects are indistinguishable from one another, and another B objects are indistinguishable from one another, etc., then the number of possible arrangements is: $\frac{N!}{A!\,B!\,C!\ldots}$
- Intro to Probability. Go over yesterday's hanging question, leading to $P = \frac{\text{Total \# Possible Successes}}{\text{Total \# of Possibilities}}$
  Answer can be expressed as $^2/_5$, 0.4, or 40%.

- Group Work: **Going to a Restaurant**
  Have the students solve these problems:
  1. There are 5 choices for an appetizer, 10 choices for a main course, and 3 choices for dessert. How many different 3-course meals can a person order?
  2. How many possible ways can four people order main courses (without appetizers or desserts)?
  3. How many possible ways can four people order different main courses?
  4. How many possible ways can four people order three different main courses and share them?
  5. On a shelf in the restaurant, there are 4 identical art books, 3 identical math books, and 5 identical history books. How many possible ways can these books be arranged on the shelf?
  6. If 3 out of the 10 main courses are vegetarian, what is the probability of you getting a random main course and having it turn out to be vegetarian?
  7. If 2 out of the 10 main courses are gluten-free, what is the probability of you getting a random main course and having it turn out to be vegetarian, *and* your friend getting a random main course and having it turn out to be gluten-free? (Should be left as a hanging question for tomorrow.)

  Solutions:
  1) 5x10x3 = 150 2) 10^4 = 10,000 3) ${}_{10}P_4 = 5040$ 4) ${}_{10}C_4 = 210$ 5) 12!/4!3!5! = 27,720
  6) 30% 7) 0.3 x 0.2 = 0.06 = 6%

Day #7

- Review.
  - What does the formula $P = \frac{\text{Total \# Possible Successes}}{\text{Total \# of Possibilities}}$ assume? Ans: that each possibility is equally likely.
    - Example: When shooting a basketball, the two possibilities are either you make a basket, or not. It is not correct to then say that you have a 50% chance to make it. Likewise, there are 11 possible outcomes (2 through 12) for rolling two dice, but you can't say that each outcome is has a 1/11 probability because they are not all equally likely.
  - Go over *Restaurant problems* from yesterday, including the last one as a hanging question.
- Rolling Two Dice. Each student should roll a pair of dice 50 times and record the number of occurrences of each value that appears. We then use computers to fill out the spreadsheet found at this link: https://www.jamieyorkpress.com/wp-content/uploads/2018/05/Dice-Experiment-Student-Data-Blank.xls This is a good opportunity to teach the students how to use formulas in spreadsheets.
- ML Book Essay: none
- Hanging questions: What happens to the average roll as you roll the pair of dice more and more?

Day #8

- Rolling Two Dice. Review yesterday. Discuss. Answer hanging question.
  - The teacher should also create a new "class results" Excel spreadsheet where each column of this new sheet is taken from the last column of each student's results. Then the last column of the new "class results" sheet should be a weighted average (because some students probably did more than 50 rolls) of all of the students results.
- A second method for finding probability.

  If A and B are independent events, the probability that both A and B will occur is the product of probabilities of each occurring separately. P(A and B) = P(A)·P(B)
- Practice. Go over more problems from *Possibility and Probability Practice Sheet.*
- ML Book Essay (due in two days): Rolling two Dice. The Spreadsheet should be embedded in the essay.

Day #9

- <u>Rolling Two Dice</u>. Calculating the probabilities of each roll.
  - In order to determine the probability of getting a certain roll, we need to first answer this question:
    How many possible outcomes are there for rolling two dice? Here are three perspectives for answering that question:
    1) There are 11 possible totals: 2 through 12.
    2) There are 21 possible combinations for the two dice: e.g., 2,5 and 5,2 <u>aren't</u> both listed.
    3) There are 36 possible permutations for the two dice: e.g., 2,5 and 5,2 <u>are</u> both listed.

    Important point: Although each of the above perspectives is valid, the first two are not helpful for determining probabilities because each of the possibilities is not equally likely.

    Additional Question: If we roll two dice 500 times, how many times would we except to roll a sum of 9?

<u>Theoretical Probabilities for Rolling Two Dice</u>
(The right-most column can be added to the Excel spreadsheet from two days ago.)

total roll	# successes	Probability
2	1	2.777777778
3	2	5.555555556
4	3	8.333333333
5	4	11.11111111
6	5	13.88888889
7	6	16.66666667
8	5	13.88888889
9	4	11.11111111
10	3	8.333333333
11	2	5.555555556
12	1	2.777777778

- <u>Practice</u>. Finish going over *Possibility and Probability Practice Sheet.*
- <u>Pascal's Triangle</u>. Just write it on the board without saying anything, and ask what patterns they notice.
- <u>Hanging questions</u>: What does Pascal's Triangle have to do with what we have studied? What patterns do you notice with Pascal's triangle?

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1

Day #10

- Go over any other hanging questions that have yet to be answered:
  - How many times more seating charts are there if we add 5 more students to the class?
    - It isn't 5! times more. For n=13, it is 1,028,160 (=18•17•16•15•14) times more.
  - If you rearranged the class into a different seating chart every second, how long would it take before you exhausted every possibility? <u>Ans</u>: For a group of size n, we get:

    n! seconds÷3600sec/hour÷24hr/day÷365days/year

    If n=13, n!=6,227,020,800 $t \approx 197$ years.

    If n=18, $n! \approx 6.40 \cdot 10^{15}$ $t \approx 203{,}000{,}000$ years

    If n=25, $n! \approx 1.55 \cdot 10^{25}$ $t \approx 492{,}000{,}000{,}000{,}000{,}000$ years
  - <u>5x6 Street Grid problem</u>.
    - Point out that the Street Grid problem is found within Pascal's Triangle – e.g., the number of possible routes with a 3x2 grid is ${}_5C_3$, and with a 6x5 grid it is ${}_{11}C_6$. Go through the logic of why this is true.
      - To get to any point (x) in the middle of the grid, you have to come either from the point above x, or the point to the left of x. Therefore the number of possible (shortest) routes to x must be the sum of the number of routes to get to those two neighboring points.
    - A new view of the Committee problem. Have 11 kids line up and hand 5 of them an "S" (meaning that they are <u>S</u>elected to be in the committee) and the other 6 an "E" (meaning that they are <u>E</u>xcluded). You have now shown one way to select a committee of 5 from 11 people. *We now see that this is the same as rearranging the letters SSSSSEEEEEE*.
    - With a 6x5 street grid, one route is to travel first 6 blocks east, and then 5 blocks south. This can be represented as EEEEEESSSSS. Therefore each possible route is simply as rearrangement of the string of letters EEEEEESSSSS.
    - Amazingly, we now see how rearranging letters, the committee problem, the street grid problem, and Pascal's triangle are all related.
  - <u>Extra Credit</u>. Essay on the 5x6 Street Grid Problem (revisited).

- <u>Pascal's Triangle</u>. Have students point out various patterns that they noticed. Here are some:
  - Each entry is the sum of the two numbers above it.
  - The sum of the n^{th} row (where n starts at 0) is 2^n.
  - The r^{th} entry of the n^{th} row is equal to ${}_nC_r$.
  - *The Binomial Theorem.* The n^{th} row of Pascal's triangle is the coefficients in the expansion of the binomial $(x+y)^n$.
  - The "hockey stick" trick: Starting at any 1, move along a diagonal, while adding up all of the numbers. Stop at any point. The ending sum is equal to the number at a 90 degree turn from where you stopped.
  - The third diagonal in is the triangular numbers: 1,3,6,10,15,21,28,36…
  - Therefore, in that same diagonal, add any two consecutive numbers and you get a square number.
  - The Fibanacci sequence is found in the shallow diagonals.

- <u>ML Book Essay (due in two days)</u>: Pascal's triangle.

Day #11

- Go over B-day problem.
  - Answers (prob that at least 2 people will have a common b-day), for various group sizes, are as follows:
    - For 5 people: 2.7%
    - For 10 people: 11.7%
    - For 15 people: 25.3%
    - For 20 people: 41.1%
    - **For 23 people: 50.7%**
    - For 25 people: 56.9%
    - For 30 people: 70.6%
    - For 40 people: 89.1%
    - For 50 people: 97.0%
    - For 70 people: 99.92%
    - For 100 people: 99.99997%
- ML Book Essay: Birthday Problem.
- Probability Summary Page:

The Probability of an Event

The probability of an event successfully occurring, P(E), *is equal to the number of possible (equally likely) success outcomes divided by the total number of possible (equally likely) outcomes.*

$$P(E) = \frac{\text{number of successful outcomes}}{\text{number of total possible outcomes}}$$

Example: One card is drawn from a standard 52-card deck. Find the probability of getting a spade.
Solution: The probability is $\frac{13}{52}$ or **25%**.

Statistical Probability

The more you repeat an event, the closer the average outcome gets to the mean outcome.

Example: If we flip a coin n times, we expect that the number of heads will get closer to 50% for larger values of n.

Example: Since the probability of rolling a sum of a 9 with two dice is 11.1%, we expect that if we roll two dice a large number of times, then we will get a 9 about 11.1% of the time.

Two Independent Events

If A and B are independent events, the probability that both A and B will occur is the product of probabilities of each occurring separately.

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

Example: If you roll a die and flip a coin, what is the probability of getting a 2 and a head?
Solution: $\frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12}$

The Probability of a Complement

If A is the complement of B, then the sum of their probabilities is equal to one, or 100%.

Example: What is the probability of flipping three coins and getting at least one head?
Solution: The complement (or opposite) of this is getting no heads, which has a probability of $\frac{1}{8}$. Therefore, the probability of getting at least one head is $1 - \frac{1}{8} = \frac{7}{8}$.

Day #12

- Give out Do *Possibility and Probability Review Sheet*
- ML Book: Finish ML Book, including title page, table of contents, and introduction.

Day #13

- Review for tomorrow's test.