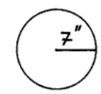
Summary of Key Concepts

Mensuration Unit (Areas and Volumes)

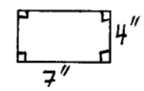
- Formulas and understanding. As always, but especially in this unit, we want to minimize the use of formulas. With every formula (except for Heron's) we should not just be blindly plugging in values. Each of these formulas has been explained or derived in class. Many math books (as well as the students' planners) include as many as 20 formulas for this topic. Students should *not* attempt to memorize and know how to use all of these formulas. This is the opposite of what I am encouraging the students to do. It is best for the students to understand the basis behind what is needed to solve each problem.

- What is π? It's more than just 3.14. The ratio of the circumference to the diameter of any circle is π:1. How many times longer is the circumference than the diameter? The answer is π.

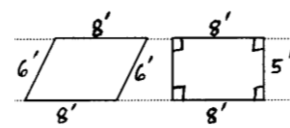
- The circumference of a circle. *The main idea*: With any circle, the circumference is π times longer than its diameter. With the circle shown here, the diameter is 14in. Therefore the circumference is $14\,\pi \approx 44.0$ inches. *Optional Formula*: $C = \pi \cdot D$



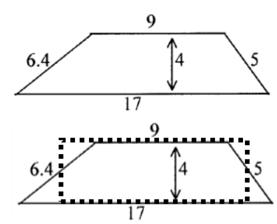
- The area of a rectangle. *The main idea*: Picture how we can draw small unit squares inside the rectangle. The area of the rectangle shown here is 28 square inches because we can fit 28 squares inside the rectangle, where each square is one square inch. *Optional Formula*: $A = B \cdot H$



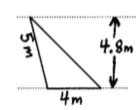
- The area of a parallelogram. *The main idea*: Picture using the "shear and stretch" to transform the parallelogram into a rectangle. The length of the edge of the parallelogram doesn't matter; only the height and base matter. The area of the parallelogram shown here is exactly the same as the rectangle: 40 ft^2. *Optional Formula*: $A = B \cdot H$



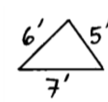
- The area of a trapezoid. *The main idea*: A trapezoid can transform into a rectangle of equal area as shown here. The rectangle has the same height of the trapezoid, and the length (or base) of the rectangle is the average of the top and bottom of the trapezoid. With the rectangle shown here, the length is 13 (the average of 9 and 17), which gives us an area of $4 \cdot 13 = 52$. *Optional Formula*: $A = \frac{1}{2}(B+T) \cdot H$



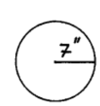
- The area of a triangle (where we can find out the height). *The main idea*: A right triangle is half a rectangle, and a non-right triangle is half a parallelogram. For area, the sides of the triangle don't matter; only the base and height matter. This triangle is half a parallelogram, which in turn can be transformed into a rectangle with an area of 19.2 m^2 ($= 4 \cdot 4.8$) The triangle's area is half as much, 9.6m^2. Sometimes we need to use the Pythagorean Theorem to calculate the height. *Optional Formula*: $A = \frac{1}{2}\, B \cdot H$



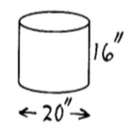
- Heron's formula for the area of a triangle (where we can't find out the height). *The main idea*: This is an instance of where we simply plug numbers into a formula. The formula is $A = \sqrt{s(s-a)(s-b)(s-c)}$, where a,b,c are the sides of the triangle, and s is the semi-perimeter (i.e., half the length of the perimeter). With the triangle shown here, the semi-perimeter is equal to 9. Putting all the numbers into the formula, we get: Area $= \sqrt{9(9-7)(9-6)(9-5)}$, which is $\sqrt{9 \cdot 2 \cdot 3 \cdot 4}$, and then $\sqrt{216} \approx 14.6$ ft^2.
Required Formula: $A = \sqrt{s(s-a)(s-b)(s-c)}$ (Don't memorize; I will give this formula on the test.)



- The area of a circle. *The main idea*: We need to use the formula, which tells us that in order to get the area of a circle we simply need to square the radius and multiply by π. With the circle shown here, we square 7 to get 49. So the area is $49\,\pi \approx 153.9$ in^2.
Required Formula: $A = \pi \cdot r^2$ **You need to <u>know</u> this formula.**

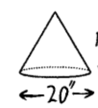


- The volume of a solid where the top equals the bottom. *The main idea*: Picture the floor of a room that is 10 feet by 15 feet. It has an area of 150 ft^2. If the room is now filled one foot deep with water, we can say that there are 150 ft^3 (cubic feet) of water. In place of water, you can also picture 150 boxes, each one a perfect cubic foot, sitting on the floor. If the room has a height of 8 feet, then we can say that its total volume is $150 \cdot 8 = 1200$ ft^3. We can picture the room filled with 1200 cubic boxes. What have we done in order to calculate this volume? We have simply multiplied the area of the floor times the height. This also applies to anything (with straight sides) where the top equals the bottom, such as the cylinder shown here. The area of the floor (base) is 100π (because the radius is 10"), and then we multiply by the height (16") to get a volume of $1600\pi \approx 5024$ in^3.



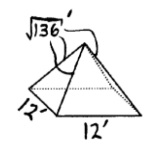
Required Formula: $V = A_{Base} \cdot H$ **You need to <u>know</u> this formula.**

- The volume of a solid where the top is a point (cone and pyramid).

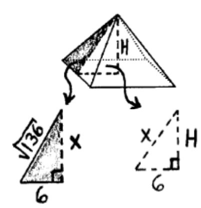


The main idea: We can use paper models to demonstrate how three "tilted" pyramids fit inside a cube. This leads us to the general (and important) conclusion that if a solid comes to a point then its volume is ⅓ of the box it fits into. With the cone shown here, we know that its volume must be ⅓ of the volume of the cylinder it fits into, which is shown in the previous example. So the volume is $\frac{1600\pi}{3} \approx 1675$ in^3.

Required Formula: $V = \frac{1}{3} A_{Base} \cdot H$ **You need to <u>know</u> this formula.**



Calculating the volume of a pyramid is more difficult if the height is not given. Using the example shown here, the length of the edge is $\sqrt{136}$. The strategy is to create two right triangles inside the pyramid, as shown below. Using the Pythagorean Theorem, we find that x (the height of one of the triangular faces) is 10, and then H (the height of the pyramid) is 8 feet. Now that we know the height, we can find the volume. The area of the base is 144 ft^2 times 8 (the height), which gives us 1152 ft^3 – the volume of the box that the pyramid fits into. One-third of this gives the volume of the pyramid = 384 ft^3.



- The volume of a sphere. *The main idea*: Archimedes discovered that the volume of a sphere is ⅔ the volume of the cylinder that it fits into. This leads to the formula for the volume of a sphere: $V = {}^4/_3 \pi r^3$. As an example, imagine a sphere with a diameter of 10 cm. Using the formula, we cube 5 (the radius) to get 125, multiply by 4 to get 500, divide by 3, and multiply by π to get the sphere's volume: $\frac{500\pi}{3} \approx 523.3$ cm^3.

Required Formula: $\mathbf{V = {}^4/_3 \pi r^3}$ **You need to <u>memorize</u> this formula.**

- Surface area (in general). *The main idea*: Simply break the solid into its parts, calculate the area of each part, and add them all together. For example, with the pyramid shown further above, we said that the height (x) of each triangle is 10 ft. Each triangle has a base of 12, a height of 10, and therefore an area of 60 ft^2. The four triangles add to an area of 240, and the pyramid's square base has an area of 144 ft^2. Finally, the total surface area (four triangles plus the square base) is 384 ft^2. (Yes, it's very coincidental that volume was 384 ft^3.) *No formula is needed.*

- Surface area of a sphere. *The main idea*: Imagine a circle on the ground, and that somehow you pull up on the center of the circle and it transforms into a hemisphere (half a sphere). The surface area of this hemisphere is twice the area of the circle it sits on. Therefore the total surface area of a full sphere is four times the area of the circle that forms its equator. For example, consider a sphere with a radius of 9 inches. The sphere's equator (circle) has an area of 81π. Therefore the surface area is four times greater than that, which is $324\pi \approx 1017$ in^2.
Required Formula: $SA = 4\pi \cdot r^2$ **You need to <u>know</u> this formula.**