A Mathematician's Apology

By G. H. HARDY

A MATHEMATICIAN, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with *ideas*. A painter makes patterns with shapes and colours, a poet with words. A painting may embody an 'idea,' but the idea is usually commonplace and unimportant. In poetry, ideas count for a good deal more; but, as Housman insisted, the importance of ideas in poetry is habitually exaggerated: 'I cannot satisfy myself that there are any such things as poetical ideas. . . . Poetry is not the thing said but a way of saying it.'

> Not all the water in the rough rude sea
> Can wash the balm from an anointed King.

Could lines be better, and could ideas be at once more trite and more false? The poverty of the ideas seems hardly to affect the beauty of the verbal pattern. A mathematician, on the other hand, has no material to work with but ideas, and so his patterns are likely to last longer, since ideas wear less with time than words.

The mathematician's patterns, like the painter's or the poet's, must be *beautiful*; the ideas, like the colours or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in the world for ugly mathematics. And here I must deal with a misconception which is still widespread (though probably much less so now than it was twenty years ago), what Whitehead has called the 'literary superstition' that love of and aesthetic appreciation of mathematics is 'a monomania confined to a few eccentrics in each generation.'

It would be difficult now to find an educated man quite insensitive to the aesthetic appeal of mathematics. It may be very hard to *define* mathematical beauty, but that is just as true of beauty of any kind—we may not know quite what we mean by a beautiful poem, but that does not prevent us from recognizing one when we read it. Even Professor Hogben, who is out to minimize at all costs the importance of the aesthetic element in mathematics, does not venture to deny its reality. 'There are, to be sure, individuals for whom mathematics exercises a coldly impersonal attraction. . . . The aesthetic appeal of mathematics may be very real for a chosen few.' But they are 'few,' he suggests, and they feel 'coldly' (and are really rather ridiculous people, who live in silly little university towns sheltered from the fresh breezes of the wide open spaces). In this he is merely echoing Whitehead's 'literary superstition.'

The fact is that there are few more 'popular' subjects than mathematics. Most people have some appreciation of mathematics, just as most people can enjoy a pleasant tune; and there are probably more people really interested in mathematics than in music. Appearances may suggest the contrary, but there are easy explanations. Music can be used to stimulate mass emotion, while mathematics cannot; and musical incapacity is recognized (no doubt rightly) as mildly discreditable, whereas most people are so frightened of the name of mathematics that they are ready, quite unaffectedly, to exaggerate their own mathematical stupidity.

A very little reflection is enough to expose the absurdity of the 'literary superstition.' There are masses of chess-players in every civilized country —in Russia, almost the whole educated population; and every chess-player can recognize and appreciate a 'beautiful' game or problem. Yet a chess problem is *simply* an exercise in pure mathematics (a game not entirely, since psychology also plays a part), and everyone who calls a problem 'beautiful' is applauding mathematical beauty, even if it is beauty of a comparatively lowly kind. Chess problems are the hymn-tunes of mathematics.

We may learn the same lesson, at a lower level but for a wider public, from bridge, or descending further, from the puzzle columns of the popular newspapers. Nearly all their immense popularity is a tribute to the drawing power of rudimentary mathematics, and the better makers of puzzles, such as Dudeney or 'Caliban,' use very little else. They know their business: what the public wants is a little intellectual 'kick,' and nothing else has quite the kick of mathematics.

I might add that there is nothing in the world which pleases even famous men (and men who have used disparaging language about mathematics) quite so much as to discover, or rediscover, a genuine mathematical theorem. Herbert Spencer republished in his autobiography a theorem about circles which he proved when he was twenty (not knowing that it had been proved over two thousand years before by Plato). Professor Soddy is a more recent and a more striking example (but *his* theorem really is his own).[1]

[1] See his letters on the 'Hexlet' in *Nature*, vols. 137–9 (1936–7).

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A chess problem is genuine mathematics, but it is in some way 'trivial' mathematics. However ingenious and intricate, however original and surprising the moves, there is something essential lacking. Chess problems are *unimportant*. The best mathematics is *serious* as well as beautiful—'important' if you like, but the word is very ambiguous, and 'serious' expresses what I mean much better.

I am not thinking of the 'practical' consequences of mathematics. I have to return to that point later: at present I will say only that if a chess problem is, in the crude sense, 'useless,' then that is equally true of most of the best mathematics; that very little of mathematics is useful practically, and that that little is comparatively dull. The 'seriousness' of a mathematical theorem lies, not in its practical consequences, which are usually negligible, but in the *significance* of the mathematical ideas which it connects. We may say, roughly, that a mathematical idea is 'significant' if it can be connected, in a natural and illuminating way, with a large complex of other mathematical ideas. Thus a serious mathematical theorem, a theorem which connects significant ideas, is likely to lead to important advances in mathematics itself and even in other sciences. No chess problem has ever affected the general development of scientific thought: Pythagoras, Newton, Einstein have in their times changed its whole direction.

The seriousness of a theorem, of course, does not *lie in* its consequences, which are merely the *evidence* for its seriousness. Shakespeare had an enormous influence on the development of the English language, Otway next to none, but that is not why Shakespeare was the better poet. He was the better poet because he wrote much better poetry. The inferiority of the chess problem, like that of Otway's poetry, lies not in its consequences but in its content.

There is one more point which I shall dismiss very shortly, not because it is uninteresting but because it is difficult, and because I have no qualifications for any serious discussion in aesthetics. The beauty of a mathematical theorem *depends* a great deal on its seriousness, as even in poetry the beauty of a line may depend to some extent on the significance of the ideas which it contains. I quoted two lines of Shakespeare as an example of the sheer beauty of a verbal pattern; but

After life's fitful fever he sleeps well

seems still more beautiful. The pattern is just as fine, and in this case the ideas have significance and the thesis is sound, so that our emotions are stirred much more deeply. The ideas do matter to the pattern, even in poetry, and much more, naturally, in mathematics; but I must not try to argue the question seriously.

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It will be clear by now that, if we are to have any chance of making progress, I must produce examples of 'real' mathematical theorems, theorems which every mathematician will admit to be first-rate. And here I am very heavily handicapped by the restrictions under which I am writing. On the one hand my examples must be very simple, and intelligible to a reader who has no specialized mathematical knowledge; no elaborate preliminary explanations must be needed; and a reader must be able to follow the proofs as well as the enunciations. These conditions exclude, for instance, many of the most beautiful theorems of the theory of numbers, such as Fermat's 'two square' theorem or the law of quadratic reciprocity. And on the other hand my examples should be drawn from 'pukka' mathematics, the mathematics of the working professional mathematician; and this condition excludes a good deal which it would be comparatively easy to make intelligible but which trespasses on logic and mathematical philosophy.

I can hardly do better than go back to the Greeks. I will state and prove two of the famous theorems of Greek mathematics. They are 'simple' theorems, simple both in idea and in execution, but there is no doubt at all about their being theorems of the highest class. Each is as fresh and significant as when it was discovered—two thousand years have not written a wrinkle on either of them. Finally, both the statements and the proofs can be mastered in an hour by any intelligent reader, however slender his mathematical equipment.

1. The first is Euclid's [2] proof of the existence of an infinity of prime numbers.

The *prime numbers* or *primes* are the numbers

(A) $$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, \ldots$$

which cannot be resolved into smaller factors.[3] Thus 37 and 317 are prime. The primes are the material out of which all numbers are built up by multiplication: thus $666 = 2.3.3.37$. Every number which is not prime itself is divisible by at least one prime (usually, of course, by several). We have to prove that there are infinitely many primes, i.e., that the series (A) never comes to an end.

Let us suppose that it does, and that

$$2, 3, 5, \ldots, P$$

is the complete series (so that P is the largest prime); and let us, on this hypothesis, consider the number Q defined by the formula

$$Q = (2.3.5.\ \ldots\ .P) + 1.$$

[2] *Elements* IX 20. The real origin of many theorems in the *Elements* is obscure, but there seems to be no particular reason for supposing that this one is not Euclid's own.

[3] There are technical reasons for not counting 1 as a prime.

It is plain that Q is not divisible by any of $2, 3, 5, \ldots, P$; for it leaves the remainder 1 when divided by any one of these numbers. But, if not itself prime, it is divisible by *some* prime, and therefore there is a prime (which may be Q itself) greater than any of them. This contradicts our hypothesis, that there is no prime greater than P; and therefore this hypothesis is false.

The proof is by *reductio ad absurdum*, and *reductio ad absurdum*, which Euclid loved so much, is one of a mathematician's finest weapons.[4] It is a far finer gambit than any chess gambit: a chess player may offer the sacrifice of a pawn or even a piece, but a mathematician offers *the game*.

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2. My second example is Pythagoras's [5] proof of the 'irrationality' of $\sqrt{2}$.

A 'rational number' is a fraction $\dfrac{a}{b}$, where a and b are integers: we may suppose that a and b have no common factor, since if they had we could remove it. To say that '$\sqrt{2}$ is irrational' is merely another way of saying that 2 cannot be expressed in the form $\left(\dfrac{a}{b}\right)^2$; and this is the same thing as saying that the equation

(B) $$a^2 = 2b^2$$

cannot be satisfied by integral values of a and b which have no common factor. This is a theorem of pure arithmetic, which does not demand any knowledge of 'irrational numbers' or depend on any theory about their nature.

We argue again by *reductio ad absurdum*; we suppose that (B) is true, a and b being integers without any common factor. It follows from (B) that a^2 is even (since $2b^2$ is divisible by 2), and therefore that a is even (since the square of an odd number is odd). If a is even then

(C) $$a = 2c$$

for some integral value of c; and therefore

$$2b^2 = a^2 = (2c)^2 = 4c^2$$

or

(D) $$b^2 = 2c^2.$$

[4] The proof can be arranged so as to avoid a *reductio*, and logicians of some schools would prefer that it should be.

[5] The proof traditionally ascribed to Pythagoras, and certainly a product of his school. The theorem occurs, in a much more general form, in Euclid (*Elements* x 9).

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Hence b^2 is even, and therefore (for the same reason as before) b is even. That is to say, a and b are both even, and so have the common factor 2. This contradicts our hypothesis, and therefore the hypothesis is false.

It follows from Pythagoras's theorem that the diagonal of a square is incommensurable with the side (that their ratio is not a rational number, that there is no unit of which both are integral multiples). For if we take the side as our unit of length, and the length of the diagonal is d, then, by a very familiar theorem also ascribed to Pythagoras,[6]

$$d^2 = 1^2 + 1^2 = 2,$$

so that d cannot be a rational number.

I could quote any number of fine theorems from the theory of numbers whose *meaning* anyone can understand. For example, there is what is called 'the fundamental theorem of arithmetic,' that any integer can be resolved, *in one way only*, into a product of primes. Thus $666 = 2.3.3.37$, and htere is no other decomposition; it is impossible that $666 = 2.11.29$ or that $13.89 = 17.73$ (and we can see so without working out the products). This theorem is, as its name implies, the foundation of higher arithmetic; but the proof, although not 'difficult,' requires a certain amount of preface and might be found tedious by an unmathematical reader.

Another famous and beautiful theorem is Fermat's 'two square' theorem. The primes may (if we ignore the special prime 2) be arranged in two classes; the primes

$$5, 13, 17, 29, 37, 41, \ldots$$

which leave remainder 1 when divided by 4, and the primes

$$3, 7, 11, 19, 23, 31, \ldots$$

which leave remainder 3. All the primes of the first class, and none of the second, can be expressed as the sum of two integral squares: thus

$$5 = 1^2 + 2^2, \ 13 = 2^2 + 3^2,$$
$$17 = 1^2 + 4^2, \ 29 = 2^2 + 5^2;$$

but 3, 7, 11, and 19 are not expressible in this way (as the reader may check by trial). This is Fermat's theorem, which is ranked, very justly, as one of the finest of arithmetic. Unfortunately there is no proof within the comprehension of anybody but a fairly expert mathematician.

There are also beautiful theorems in the 'theory of aggregates' (*Mengenlehre*), such as Cantor's theorem of the 'non-enumerability' of the continuum. Here there is just the opposite difficulty. The proof is easy enough, when once the language has been mastered, but considerable explanation is necessary before the *meaning* of the theorem becomes clear.

[6] Euclid, *Elements* I 47.

So I will not try to give more examples. Those which I have given are test cases, and a reader who cannot appreciate them is unlikely to appreciate anything in mathematics.

I said that a mathematician was a maker of patterns of ideas, and that beauty and seriousness were the criteria by which his patterns should be judged. I can hardly believe that anyone who has understood the two theorems will dispute that they pass these tests. If we compare them with Dudeney's most ingenious puzzles, or the finest chess problems that masters of that art have composed, their superiority in both respects stands out: there is an unmistakable difference of class. They are much more serious, and also much more beautiful: can we define, a little more closely, where their superiority lies?

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I will end with a summary of my conclusions, but putting them in a more personal way. I said at the beginning that anyone who defends his subject will find that he is defending himself; and my justification of the life of a professional mathematician is bound to be, at bottom, a justification of my own. Thus this concluding section will be in its substance a fragment of autobiography.

I cannot remember ever having wanted to be anything but a mathematician. I suppose that it was always clear that my specific abilities lay that way, and it never occurred to me to question the verdict of my elders. I do not remember having felt, as a boy, any *passion* for mathematics, and such notions as I may have had of the career of a mathematician were far from noble. I thought of mathematics in terms of examinations and scholarships: I wanted to beat other boys, and this seemed to be the way in which I could do so most decisively.

I was about fifteen when (in a rather odd way) my ambitions took a sharper turn. There is a book by 'Alan St. Aubyn'[9] called *A Fellow of Trinity*, one of a series dealing with what is supposed to be Cambridge college life. I suppose that it is a worse book than most of Marie Corelli's; but a book can hardly be entirely bad if it fires a clever boy's imagination. There are two heroes, a primary hero called Flowers, who is almost wholly good, and a secondary hero, a much weaker vessel, called Brown. Flowers and Brown find many dangers in university life, but the worst is a gambling saloon in Chesterton[10] run by the Misses Bellenden, two fascinating but extremely wicked young ladies. Flowers survives all these troubles, is Second Wrangler and Senior Classic, and succeeds automatically to a Fellowship (as I suppose he would have done then). Brown succumbs, ruins his parents, takes to drink, is saved from delirium tremens during a thunderstorm only by the prayers of the Junior Dean, has much difficulty in obtaining even an Ordinary Degree, and ultimately becomes a missionary. The friendship is not shattered by these unhappy events, and Flowers's thoughts stray to Brown, with affectionate pity, as he drinks port and eats walnuts for the first time in Senior Combination Room.

Now Flowers was a decent enough fellow (so far as 'Alan St. Aubyn' could draw one), but even my unsophisticated mind refused to accept him as clever. If he could do these things, why not I? In particular, the final scene in Combination Room fascinated me completely, and from that time, until I obtained one, mathematics meant to me primarily a Fellowship of Trinity.

[9] 'Alan St. Aubyn' was Mrs. Frances Marshall, wife of Matthew Marshall.
[10] Actually, Chesterton lacks picturesque features.

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I found at once, when I came to Cambridge, that a Fellowship implied 'original work,' but it was a long time before I formed any definite idea of research. I had of course found at school, as every future mathematician does, that I could often do things much better than my teachers; and even at Cambridge I found, though naturally much less frequently, that I could sometimes do things better than the College lecturers. But I was really quite ignorant, even when I took the Tripos, of the subjects on which I have spent the rest of my life; and I still thought of mathematics as essentially a 'competitive' subject. My eyes were first opened by Professor Love, who taught me for a few terms and gave me my first serious conception of analysis. But the great debt which I owe to him—he was, after all, primarily an applied mathematician—was his advice to read Jordan's famous *Cours d'analyse*; and I shall never forget the astonishment with which I read that remarkable work, the first inspiration for so many mathematicians of my generation, and learnt for the first time as I read it what mathematics really meant. From that time onwards I was in my way a real mathematician, with sound mathematical ambitions and a genuine passion for mathematics.

I wrote a great deal during the next ten years, but very little of any importance; there are not more than four or five papers which I can still remember with some satisfaction. The real crises of my career came ten or twelve years later, in 1911, when I began my long collaboration with Littlewood, and in 1913, when I discovered Ramanujan. All my best work since then has been bound up with theirs, and it is obvious that my association with them was the decisive event of my life. I still say to myself when I am depressed, and find myself forced to listen to pompous and tiresome people, 'Well, I have done one thing *you* could never have done, and that is to have collaborated with both Littlewood and Ramanujan on something like equal terms.' It is to them that I owe an unusually late maturity: I was at my best at a little past forty, when I was a professor at Oxford. Since then I have suffered from that steady deterioration which is the common fate of elderly men and particularly of elderly mathematicians. A mathematician may still be competent enough at sixty, but it is useless to expect him to have original ideas.

It is plain now that my life, for what it is worth, is finished, and that nothing I can do can perceptibly increase or diminish its value. It is very difficult to be dispassionate, but I count it a 'success'; I have had more reward and not less than was due to a man of my particular grade of ability. I have held a series of comfortable and 'dignified' positions. I have had very little trouble with the duller routine of universities. I hate 'teaching,' and have had to do very little, such teaching as I have done having been almost entirely supervision of research; I love lecturing, and have lectured a great deal to extremely able classes; and I have always had plenty of leisure for the researches which have been the one great permanent happiness of my life. I have found it easy to work with others, and have collaborated on a large scale with two exceptional mathematicians; and this has enabled me to add to mathematics a good deal more than I could reasonably have expected. I have had my disappointments, like any other mathematician, but none of them has been too serious or has made me particularly unhappy. If I had been offered a life neither better nor worse when I was twenty, I would have accepted without hesitation.

It seems absurd to suppose that I could have 'done better.' I have no linguistic or artistic ability, and very little interest in experimental science. I might have been a tolerable philosopher, but not one of a very original kind. I think that I might have made a good lawyer; but journalism is the only profession, outside academic life, in which I should have felt really confident of my chances. There is no doubt that I was right to be a mathematician, if the criterion is to be what is commonly called success.

My choice was right, then, if what I wanted was a reasonably comfortable and happy life. But solicitors and stockbrokers and bookmakers often lead comfortable and happy lives, and it is very difficult to see how the world is the richer for their existence. Is there any sense in which I can claim that my life has been less futile than theirs? It seems to me again that there is only one possible answer: yes, perhaps, but, if so, for one reason only.

I have never done anything 'useful.' No discovery of mine has made, or is likely to make, directly or indirectly, for good or ill, the least difference to the amenity of the world. I have helped to train other mathematicians, but mathematicians of the same kind as myself, and their work has been, so far at any rate as I have helped them to it, as useless as my own. Judged by all practical standards, the value of my mathematical life is nil; and outside mathematics it is trivial anyhow. I have just one chance of escaping a verdict of completely triviality, that I may be judged to have created something worth creating. And that I have created something is undeniable: the question is about its value.

The case for my life, then, or for that of any one else who has been a mathematician in the same sense in which I have been one, is this: that I have added something to knowledge, and helped others to add more; and that these somethings have a value which differs in degree only, and not in kind, from that of the creations of the great mathematicians, or of any of the other artists, great or small, who have left some kind of memorial behind them.

1 Mathematics as an Art

By JOHN WILLIAM NAVIN SULLIVAN

THE prestige enjoyed by mathematicians in every civilized country is not altogether easy to understand. Anything which is valued by the generality of men is either useful or pleasant, or both. Farming is a valued occupation, and so is piano-playing, but why are the activities of the mathematician considered to be important? It might be said that mathematics is valued for its applications. Everybody knows that modern civilization depends, to an unprecedented extent, upon science, and a great deal of that science would be impossible in the absence of a highly developed mathematical technique. This is doubtless a weighty consideration; and it is true that even mathematics has benefited by the increased esteem in which science is held as a consequence of the magnificent murderous capacities it exhibited in the late war. But it is doubtful whether this consideration alone is adequate to explain the exalted position accorded to mathematics throughout a larger part of its history. On the other hand, it does not seem as if we could attach much importance to the claim made by many mathematicians that their science is a delightful art. Their claim is doubtless justified; but the fact that a few, a very few, unusual individuals obtain great pleasure from some incomprehensible pursuit is no reason why the ordinary man should admire them and support them. Chess professorships are not established, but there are probably more people who appreciate the "beauties" of chess than appreciate the beauties of mathematics. The present position accorded to mathematics by the non-mathematical public is due partly to the usefulness of mathematics and partly to the persistence, in a more or less vague form, of old and erroneous ideas respecting its real significance. It is only within quite recent times, indeed, that the correct status of mathematics has been discovered, although there are many and very important aspects of this wonderful activity which still remain mysterious.

It is probable that mathematics originated with Pythagoras. There is no clear evidence that that distinctive activity we call mathematical reasoning was fully recognised and practised by any one before Pythagoras. Certain arithmetical results had long been known, of course, but neither geometry or algebra had been created. The geometrical formulas used by the ancient Egyptians, for example, deal chiefly with land-surveying problems, and were evidently obtained empirically. They are usually wrong and are nowhere accompanied by proofs. It seems strange that this particular possibility of the mind should have been discovered so late, for it is completely independent of external circumstances. Even music, the most independent of what are usually classed as the arts, is more dependent on its *milieu* than is mathematics. Nevertheless, both music and mathematics, the two most "subjective" of human creations, have been singularly late and slow in their development. And just as it is impossible for us to understand what their rudimentary music meant to the Greeks, so it is impossible to enter into the difficulties of the pre-mathematical mind. The musical enthusiasms of Plato are just as remote from us as are the difficulties of that Chinese Emperor who could not be convinced by the abstract proof that the volume of a sphere varies as the cube of its radius. He had various sized spheres made, filled with water, and weighed. This was his conception of a proof. And this must have been typical of the ancient mind. They lacked a faculty, just as the Greeks lacked a harmonic sense.

It is not surprising, therefore, that when the mind first became aware of this unsuspected power it did not undersand its true nature. It appeared vastly more significant—or at least significant in a different way from what it really is. To the Pythagoreans, overwhelmed by the æsthetic charm of the theorems they discovered, number became the principle of all things. Number was supposed to be the very essence of the real; other things that could be predicted of the real were merely aspects of number. Thus the number one is what, in a certain aspect, we call reason, for reason is unchangeable, and the very essence of unchangeableness is expressed by the number one: the number two, on the other hand, is unlimited and indeterminate; "opinion" as contrasted with "reason," is an expression of the number two: again, the proper essence of marriage is expressed by the number five, since five is reached by combining three and two, that is, the first masculine with the first feminine number: the number four is the essence of justice, for four is the product of equals. To understand this outlook it is only necessary to enter into that condition of mind which takes any analogy to represent a real bond. Thus odd and even, male and female, light and darkness, straight and curved, all become expressions of some profound principle of opposition which informs the world. There are many mystical and semi-mystical writers of the present day who find themselves able to think in this manner; and it must be admitted that there is a not uncommon type of mind, otherwise orthodox, which is able to adopt this kind of reasoning without discomfort. Even Goethe, in his *Farbenlehre*, finds that a triangle has a mystic significance.

As long as the true logical status of mathematical propositions remained

unknown it was possible for many mathematicians to surmise that they must have some profound relation to the structure of the universe. Mathematical propositions were supposed to be true quite independently of our minds, and from this fact the existence of God was deduced. This doctrine was, indeed, a refinement on the Pythagorean fantasies, and was held by many who did not believe in the mystic properties of numbers. But the mystical outlook on numbers continued to flourish for many centuries. Thus St. Augustine, speaking of the perfection of the number six, says:—

> Six is a number perfect in itself, and not because God created all things in six days; rather the inverse is true, that God created all things in six days because this number is perfect, and it would remain perfect, even if the work of the six days did not exist.

From speculations of this sort the Pythagorean doctrine developed, on the one hand, in a thoroughly respectable philosophic manner into the doctrine of necessary truths, and on the other descended to cabbalistic imbecilities. Even very good mathematicians became cabbalists. The famous Michael Stifel, one of the most celebrated algebraists of the sixteenth century, considered that by far the most important part of his work was his cabbalistic interpretation of the prophetic books of the Bible. That this method enjoyed a high prestige is sufficiently shown by the general belief accorded to his prophecy that the world would come to an end on October 3, 1533—with the result that a large number of people abandoned their occupations and wasted their substance, to find, when the date came and passed, that they were ruined. Such geometric figures as star-polygons, also, were supposed to be of profound significance; and even Kepler, after demonstrating their mathematical properties with perfect rigour, goes on to explain their use as amulets or conjurations. As another instance of the persistence of this way of regarding mathematical entities it may be mentioned that the early development of infinite series was positively hampered by the exaggerated significance attached to mathematical operations. Thus in the time of Leibnitz it was believed that the sum of an infinite number of zeros was equal to ½; and it was attempted to make this obvious idiocy plausible by saying that it was the mathematical analogue of the creation of the world out of nothing.

There is sufficient evidence, then, that there has existed a widespread tendency to attribute a mystic significance to mathematical entities. And there are many indications, even at the present day, that this tendency persists. It is probable, then, that the prestige enjoyed by the mathematician is not altogether unconnected with the prestige enjoyed by any master of the occult. The position accorded to the mathematician has been, to some extent, due to the superstitions of mankind, although doubtless it can be justified on rational grounds. For a long time, particularly in India and Arabia, men became mathematicians to become astronomers,

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and they became astronomers to become astrologers. The aim of their activities was superstition, not science. And even in Europe, and for some years after the beginning of the Renaissance, astrology and kindred subjects were important justifications of mathematical researches. We no longer believe in astrology or mystic hexagons and the like; but nobody who is acquainted with some of the imaginative but non-scientific people can help suspecting that Pythagoreanism is not yet dead.

When we come to consider the other justification of mathematics derived from the Pythagorean outlook—its justification on the ground that it provided the clearest and most indubitable examples of necessary truths—we find this outlook, so far from being extinct, still taught by eminent professors of logic. Yet the non-Euclidean geometries, now a century old, have made it quite untenable. The point of view is well expressed by Descartes in a famous passage from his Fifth Meditation:—

J'imagine un triangle, encore qu'il n'y ait peut-être en aucun lieu du monde hors de ma pensée une telle figure et qu'il n'y en ait jamais eu, il ne laisse pas néanmoins d'y avoir une certaine nature ou forme, ou essence déterminée de cette figure, laquelle est immuable et éternelle, que je n'ai point inventée et qui ne dépend en aucune façon de mon esprit; comme il paraît, de ce que l'on peut démontrer diverses propriétés de ce triangle, à savoir que ses trois angles sont égaux à deux droits, que le plus grand angle est soutenu par le plus grand côté, et autres semblables, lesquelles maintenant, soit que je le veuille ou non, je reconnais très clairement et très évidemment être en lui, encore que je n'y aie pensé auparavant en aucune façon, lorsque je me suis imaginé la première fois un triangle, et, pourtant, on ne peut pas dire que je les ai feintes ni inventées.

A triangle, therefore, according to Descartes, does not depend in any way upon one's mind. It has an eternal and immutable existence quite independent of our knowledge of it. Its properties are discovered by our minds, but do not in any way depend upon them. This way of regarding geometrical entities lasted for two thousand years. To the Platonists geometrical propositions, expressing eternal truths, are concerned with the world of Ideas, a world apart, separate from the sensible world. To the followers of St. Augustine these Platonic Ideas became the ideas of God; and to the followers of St. Thomas Aquinas they became aspects of the Divine Word. Throughout the whole of scholastic philosophy the necessary truth of geometrical propositions played a very important part; and, as we have said, there are certain philosophers of the present day who regard the axioms of Euclid's geometry as unescapable truths. If this outlook be justified, then the mathematical faculty gives us access, as it were, to an eternally existing, although not sensible, world. Before the discovery of mathematics this world was unknown to us, but it nevertheless existed, and Pythagoras no more invented mathematics than Columbus invented America. Is this a true description of the nature of mathematics? Is mathematics really a body of knowledge about an existing, but super-

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sensible, world? Some of us will be reminded of the claims certain theorists have made for music. Some musicians have been so impressed by the extraordinary impression of *inevitability* given by certain musical works that they have declared that there must be a kind of heaven in which musical phrases already exist. The great musician discovers these phrases—he hears them, as it were. Inferior musicians hear them imperfectly; they give a confused and distorted rendering of the pure and celestial reality. The faculty for grasping celestial music is rare; the faculty for grasping celestial triangles, on the other hand, seems to be possessed by all men.

These notions, so far as geometry is concerned, rest upon the supposed necessity of Euclid's axioms. The fundamental postulates of Euclidean geometry were regarded, up to the early part of the nineteenth century, by practically every mathematician and philosopher, as necessities of thought. It was not only that Euclidean geometry was considered to be the geometry of existing space—it was the necessary geometry of any space. Yet it had quite early been realized that there was a fault in this apparently impeccable edifice. The well-known definition of parallel lines was not, it was felt, sufficiently obvious, and the Greek followers of Euclid made attempts to improve it. The Arabians also, when they acquired the Greek mathematics, found the parallel axiom unsatisfactory. No one doubted that this was a necessary truth, but they thought there should be some way of deducing it from the other and simpler axioms of Euclid. With the spread of mathematics in Europe came a whole host of attempted demonstrations of the parallel axiom. Some of these were miracles of ingenuity, but it could be shown in every case that they rested on assumptions which were equivalent to accepting the parallel axiom itself. One of the most noteworthy of these investigations was that of the Jesuit priest Girolamo Saccheri, whose treatise appeared early in the eighteenth century. Saccheri was an extremely able logician, too able to make unjustified assumptions. His method was to develop the consequences of denying Euclid's parallel axiom while retaining all the others. In this way he expected to develop a geometry which should be self-contradictory, since he had no doubt that the parallel axiom was a necessary truth. But although Saccheri struggled very hard he did not succeed in contradicting himself; what he actually did was to lay the foundations of the first non-Euclidean geometry. But even so, and although D'Alembert was expressing the opinion of all the mathematicians of his time in declaring the parallel axiom to be the "scandal" of geometry, no one seems seriously to have doubted it. It appears that the first mathematician to realize that the parallel axiom could be denied and yet a perfectly self-consistent geometry constructed was Gauss. But Gauss quite realized how staggering, how shocking, a thing he had done, and was afraid to publish his researches. It was re-served for a Russian, Lobachevsky, and a Hungarian, Bolyai, to publish the first non-Euclidean geometry. It at once became obvious that Euclid's axioms were not necessities of thought, but something quite different, and that there was no reason to suppose that triangles had any celestial existence whatever.

The further development of non-Euclidean geometry and its application to physical phenomena by Einstein have shown that Euclid's geometry is not only not a necessity of thought but is not even the most convenient geometry to apply to existing space. And with this there has come, of course, a profound change in the status we ascribe to mathematical entities, and a different estimate of the significance of the mathematician's activities. We can start from any set of axioms we please, provided they are consistent with themselves and one another, and work out the logical consequences of them. By doing so we create a branch of mathematics. The primary definitions and postulates are not given by experience, nor are they necessities of thought. The mathematician is entirely free, within the limits of his imagination, to construct what worlds he pleases. What he is to imagine is a matter for his own caprice; he is not thereby discovering the fundamental principles of the universe nor becoming acquainted with the ideas of God. If he can find, in experience, sets of entities which obey the same logical scheme as his mathematical entities, then he has applied his mathematics to the external world; he has created a branch of science. Why the external world should obey the laws of logic, why, in fact, science should be possible, is not at all an easy question to answer. There are even indications in modern physical theories which make some men of science doubt whether the universe will turn out to be finally rational. But, however that may be, there is certainly no more reason to suppose that natural phenomena must obey any particular geometry than there is to suppose that the music of the spheres, should we ever hear it, must be in the diatonic scale.

Since, then, mathematics is an entirely free activity, unconditioned by the external world, it is more just to call it an art than a science. It is as independent as music of the external world; and although, unlike music, it can be used to illuminate natural phenomena, it is just as "subjective," just as much of a product of the free creative imagination. And it is not at all difficult to discover that the mathematicians are impelled by the same incentives and experience the same satisfactions as other artists. The literature of mathematics is full of æsthetic terms, and the mathematician who said that he was less interested in results than in the beauty of the methods by which he found the results was not expressing an unusual sentiment.

But to say that mathematics is an art is not to say that it is a mere amusement. Art is not something which exists merely to satisfy an

"æsthetic emotion." Art which is worthy of the name reveals to us some aspect of reality. This is possible because our consciousness and the external world are not two independent entities. Science has advanced sufficiently far for us to be able to say that the external world is, at least very largely, our own creation; and we understand much of what we have created by understanding the laws of our own being, the laws in accordance with which we must create. There is no reason to suppose that there is a heavenly storehouse of musical phrases, but it is true that the musician can reveal to us a reality which is profounder than that of common sense. "He who understands the meaning of my music," Beethoven is reported to have said, "shall be free from the miseries that afflict other men." We may not know what he meant, but it is evident that he regarded music as something that had meaning, something that revealed a reality which cannot normally be perceived. And it seems that the mathematician, in creating his art, is exhibiting that movement of our minds that has created the spatio-temporal material universe we know. Mathematics, as much as music or any other art, is one of the means by which we rise to a complete self-consciousness. The significance of mathematics resides precisely in the fact that it is an art; by informing us of the nature of our own minds it informs us of much that depends on our minds. It does not enable us to explore some remote region of the eternally existent; it helps to show us how far what exists depends upon the way in which we exist. We are the law-givers of the universe; it is even possible that we can experience nothing but what we have created, and that the greatest of our mathematical creations is the material universe itself.

We return thus to a sort of inverted Pythagorean outlook. Mathematics is of profound significance in the universe, not because it exhibits principles that we obey, but because it exhibits principles that we impose. It shows us the laws of our own being and the necessary conditions of experience. And is it not true that the other arts do something similar in those regions of experience which are not of the intellect alone? May it not be that the meaning Beethoven declared his music to possess is that, although man seems to live in an alien universe, yet it is true of the whole of experience as well as of that part of it which is the subject of science that what man finds is what he has created, and that the spirit of man is indeed free, eternally subject only to its own decrees? But however this may be it is certain that the real function of art is to increase our self-consciousness; to make us more aware of what we are, and therefore of what the universe in which we live really is. And since mathematics, in its own way, also performs this function, it is not only æsthetically charming but profoundly significant. It is an art, and a great art. It is on this, besides its usefulness in practical life, that its claim to esteem must be based.

See skulking Truth to her old cavern fled,
Mountains of Casuistry heap'd o'er her head!
Philosophy, that lean'd on Heav'n before,
Shrinks to her second cause, and is no more.
Physic of Metaphysic begs defence,
And Metaphysic calls for aid on Sense!
See Mystery to Mathematics fly!

—ALEXANDER POPE

Mathematics and the Metaphysicians

By BERTRAND RUSSELL*

THE nineteenth century, which prided itself upon the invention of steam and evolution, might have derived a more legitimate title to fame from the discovery of pure mathematics. This science, like most others, was baptised long before it was born; and thus we find writers before the nineteenth century alluding to what they called pure mathematics. But if they had been asked what this subject was, they would only have been able to say that it consisted of Arithmetic, Algebra, Geometry, and so on. As to what these studies had in common, and as to what distinguished them from applied mathematics, our ancestors were completely in the dark.

Pure mathematics was discovered by Boole, in a work which he called the *Laws of Thought* (1854). This work abounds in asseverations that it is not mathematical, the fact being that Boole was too modest to suppose his book the first ever written on mathematics. He was also mistaken in supposing that he was dealing with the laws of thought: the question how people actually think was quite irrelevant to him, and if his book had really contained the laws of thought, it was curious that no one should ever have thought in such a way before. His book was in fact concerned with formal logic, and this is the same thing as mathematics.

Pure mathematics consists entirely of assertions to the effect that, if such and such a proposition is true of *anything*, then such and such another proposition is true of that thing. It is essential not to discuss whether the first proposition is really true, and not to mention what the anything is, of which it is supposed to be true. Both these points would belong to applied mathematics. We start, in pure mathematics, from certain rules of inference, by which we can infer that *if* one proposition is true, then so is some other proposition. These rules of inference constitute the major part of the principles of formal logic. We then take any hypothesis that seems amusing, and deduce its consequences. *If* our hypothesis is about *anything*, and not about some one or more particular things, then our deductions constitute mathematics. Thus mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true. People who have been puzzled by the beginnings of mathematics will, I hope, find comfort in this definition, and will probably agree that it is accurate.

* For a biographical note about Bertrand Russell, see p. 377.

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As one of the chief triumphs of modern mathematics consists in having discovered what mathematics really is, a few more words on this subject may not be amiss. It is common to start any branch of mathematics—for instance, Geometry—with a certain number of primitive ideas, supposed incapable of definition, and a certain number of primitive propositions or axioms, supposed incapable of proof. Now the fact is that, though there are indefinables and indemonstrables in every branch of applied mathematics, there are none in pure mathematics except such as belong to general logic. Logic, broadly speaking, is distinguished by the fact that its propositions can be put into a form in which they apply to anything whatever. All pure mathematics—Arithmetic, Analysis, and Geometry—is built up by combinations of the primitive ideas of logic, and its propositions are deduced from the general axioms of logic, such as the syllogism and the other rules of inference. And this is no longer a dream or an aspiration. On the contrary, over the greater and more difficult part of the domain of mathematics, it has been already accomplished; in the few remaining cases, there is no special difficulty, and it is now being rapidly achieved. Philosophers have disputed for ages whether such deduction was possible; mathematicians have sat down and made the deduction. For the philosophers there is now nothing left but graceful acknowledgments.

The subject of formal logic, which has thus at last shown itself to be identical with mathematics, was, as every one knows, invented by Aristotle, and formed the chief study (other than theology) of the Middle Ages. But Aristotle never got beyond the syllogism, which is a very small part of the subject, and the schoolmen never got beyond Aristotle. If any proof were required of our superiority to the mediæval doctors, it might be found in this. Throughout the Middle Ages, almost all the best intellects devoted themselves to formal logic, whereas in the nineteenth century only an infinitesimal proportion of the world's thought went into this subject. Nevertheless, in each decade since 1850 more has been done to advance the subject than in the whole period from Aristotle to Leibniz. People have discovered how to make reasoning symbolic, as it is in Algebra, so that deductions are effected by mathematical rules. They have discovered many rules besides the syllogism, and a new branch of logic, called the Logic of Relatives,[1] has been invented to deal with topics that wholly surpassed the powers of the old logic, though they form the chief contents of mathematics.

It is not easy for the lay mind to realise the importance of symbolism in discussing the foundations of mathematics, and the explanation may perhaps seem strangely paradoxical. The fact is that symbolism is useful because it makes things difficult. (This is not true of the advanced parts of mathematics, but only of the beginnings.) What we wish to know is, what can be deduced from what. Now, in the beginnings, everything is self-evident; and it is very hard to see whether one self-evident proposition follows from another or not. Obviousness is always the enemy to correctness. Hence we invent some new and difficult symbolism, in which nothing seems obvious. Then we set up certain rules for operating on the symbols, and the whole thing becomes mechanical. In this way we find out what must be taken as premiss and what can be demonstrated or defined. For instance, the whole of Arithmetic and Algebra has been shown to require three indefinable notions and five indemonstrable propositions. But without a symbolism it would have been very hard to find this out. It is so obvious that two and two are four, that we can hardly make ourselves sufficiently sceptical to doubt whether it can be proved. And the same holds in other cases where self-evident things are to be proved.

But the proof of self-evident propositions may seem, to the uninitiated, a somewhat frivolous occupation. To this we might reply that it is often by no means self-evident that one obvious proposition follows from another obvious proposition; so that we are really discovering new truths when we prove what is evident by a method which is not evident. But a more interesting retort is, that since people have tried to prove obvious propositions, they have found that many of them are false. Self-evidence is often a mere will-o'-the-wisp, which is sure to lead us astray if we take it as our guide. For instance, nothing is plainer than that a whole always has more terms than a part, or that a number is increased by adding one to it. But these propositions are now known to be usually false. Most numbers are infinite, and if a number is infinite you may add ones to it as long as you like without disturbing it in the least. One of the merits of a proof is that it instils a certain doubt as to the result proved; and when what is obvious can be proved in some cases, but not in others, it becomes possible to suppose that in these other cases it is false.

The great master of the art of formal reasoning, among the men of our own day, is an Italian, Professor Peano, of the University of Turin.[2] He

[1] This subject is due in the main to Mr. C. S. Peirce.
[2] I ought to have added Frege, but his writings were unknown to me when this article was written. [Note added in 1917.]

has reduced the greater part of mathematics (and he or his followers will, in time, have reduced the whole) to strict symbolic form, in which there are no words at all. In the ordinary mathematical books, there are no doubt fewer words than most readers would wish. Still, little phrases occur, such as *therefore, let us assume, consider*, or *hence it follows*. All these, however, are a concession, and are swept away by Professor Peano. For instance, if we wish to learn the whole of Arithmetic, Algebra, the Calculus, and indeed all that is usually called pure mathematics (except Geometry), we must start with a dictionary of three words. One symbol stands for *zero*, another for *number*, and a third for *next after*. What these ideas mean, it is necessary to know if you wish to become an arithmetician. But after symbols have been invented for these three ideas, not another word is required in the whole development. All future symbols are symbolically explained by means of these three. Even these three can be explained by means of the notions of *relation* and *class*; but this requires the Logic of Relations, which Professor Peano has never taken up. It must be admitted that what a mathematician has to know to begin with is not much. There are at most a dozen notions out of which all the notions in all pure mathematics (including Geometry) are compounded. Professor Peano, who is assisted by a very able school of young Italian disciples, has shown how this may be done; and although the method which he has invented is capable of being carried a good deal further than he has carried it, the honour of the pioneer must belong to him.

Two hundred years ago, Leibniz foresaw the science which Peano has perfected, and endeavoured to create it. He was prevented from succeeding by respect for the authority of Aristotle, whom he could not believe guilty of definite, formal fallacies; but the subject which he desired to create now exists, in spite of the patronising contempt with which his schemes have been treated by all superior persons. From this "Universal Characteristic," as he called it, he hoped for a solution of all problems, and an end to all disputes. "If controversies were to arise," he says, "there would be no more need of disputation between two philosophers than between two accountants. For it would suffice to take their pens in their hands, to sit down to their desks, and to say to each other (with a friend as witness, if they liked), 'Let us calculate.' " This optimism has now appeared to be somewhat excessive; there still are problems whose solution is doubtful, and disputes which calculation cannot decide. But over an enormous field of what was formerly controversial, Leibniz's dream has become sober fact. In the whole philosophy of mathematics, which used to be at least as full of doubt as any other part of philosophy, order and certainty have replaced the confusion and hesitation which formerly reigned. Philosophers, of course, have not yet discovered this fact,

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and continue to write on such subjects in the old way. But mathematicians, at last in Italy, have now the power of treating the principles of mathematics in an exact and masterly manner, by means of which the certainty of mathematics extends also to mathematical philosophy. Hence many of the topics which used to be placed among the great mysteries—for example, the natures of infinity, of continuity, of space, time and motion—are now no longer in any degree open to doubt or discussion. Those who wish to know the nature of these things need only read the works of such men as Peano or Georg Cantor; they will there find exact and indubitable expositions of all these quondam mysteries.

In this capricious world, nothing is more capricious than posthumous fame. One of the most notable examples of posterity's lack of judgment is the Eleatic Zeno. This man, who may be regarded as the founder of the philosophy of infinity, appears in Plato's Parmenides in the privileged position of instructor to Socrates. He invented four arguments, all immeasurably subtle and profound, to prove that motion is impossible, that Achilles can never overtake the tortoise, and that an arrow in flight is really at rest. After being refuted by Aristotle, and by every subsequent philosopher from that day to our own, these arguments were reinstated, and made the basis of a mathematical renaissance, by a German professor, who probably never dreamed of any connection between himself and Zeno. Weierstrass,[3] by strictly banishing from mathematics the use of infinitesimals, has at last shown that we live in an unchanging world, and that the arrow in its flight is truly at rest. Zeno's only error lay in inferring (if he did infer) that, because there is no such thing as a state of change, therefore the world is in the same state at any one time as at any other. This is a consequence which by no means follows; and in this respect, the German mathematician is more constructive than the ingenious Greek. Weierstrass has been able, by embodying his views in mathematics, where familiarity with truth eliminates the vulgar prejudices of common sense, to invest Zeno's paradoxes with the respectable air of platitudes; and if the result is less delightful to the lover of reason than Zeno's bold defiance, it is at any rate more calculated to appease the mass of academic mankind.

Zeno was concerned, as a matter of fact, with three problems, each presented by motion, but each more abstract than motion, and capable of a purely arithmetical treatment. These are the problems of the infinitesimal, the infinite, and continuity. To state clearly the difficulties involved, was to accomplish perhaps the hardest part of the philosopher's task. This was done by Zeno. From him to our own day, the finest intellects of each generation in turn attacked the problems, but achieved, broadly speaking, nothing. In our own time, however, three men—Weierstrass, Dedekind, and Cantor—have not merely advanced the three problems, but have completely solved them. The solutions, for those acquainted with mathematics, are so clear as to leave no longer the slightest doubt or difficulty. This achievement is probably the greatest of which our age has to boast; and I know of no age (except perhaps the golden age of Greece) which has a more convincing proof to offer of the transcendent genius of its great men. Of the three problems, that of the infinitesimal was solved by Weierstrass; the solution of the other two was begun by Dedekind, and definitively accomplished by Cantor.

[3] Professor of Mathematics in the University of Berlin. He died in 1897.

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The infinitesimal played formerly a great part in mathematics. It was introduced by the Greeks, who regarded a circle as differing infinitesimally from a polygon with a very large number of very small equal sides. It gradually grew in importance, until, when Leibniz invented the Infinitesimal Calculus, it seemed to become the fundamental notion of all higher mathematics. Carlyle tells, in his *Frederick the Great*, how Leibniz used to discourse to Queen Sophia Charlotte of Prussia concerning the infinitely little, and how she would reply that on that subject she needed no instruction—the behaviour of courtiers had made her thoroughly familiar with it. But philosophers and mathematicians—who for the most part had less acquaintance with courts—continued to discuss this topic, though without making any advance. The Calculus required continuity, and continuity was supposed to require the infinitely little; but nobody could discover what the infinitely little might be. It was plainly not quite zero, because a sufficiently large number of infinitesimals, added together, were seen to make up a finite whole. But nobody could point out any fraction which was not zero, and yet not finite. Thus there was a deadlock. But at last Weierstrass discovered that the infinitesimal was not needed at all, and that everything could be accomplished without it. Thus there was no longer any need to suppose that there was such a thing. Nowadays, therefore, mathematicians are more dignified than Leibniz: instead of talking about the infinitely small, they talk about the infinitely great—a subject which, however appropriate to monarchs, seems, unfortunately, to interest them even less than the infinitely little interested the monarchs to whom Leibniz discoursed.

The banishment of the infinitesimal has all sorts of odd consequences, to which one has to become gradually accustomed. For example, there is no such thing as the next moment. The interval between one moment and the next would have to be infinitesimal, since, if we take two moments with a finite interval between them, there are always other moments in the interval. Thus if there are to be no infinitesimals, no two moments are quite consecutive, but there are always other moments between any two. Hence there must be an infinite number of moments between any two; because if there were a finite number one would be nearest the first of the two moments, and therefore next to it. This might be thought to be a difficulty; but, as a matter of fact, it is here that the philosophy of the infinite comes in, and makes all straight.

The same sort of thing happens in space. If any piece of matter be cut in two, and then each part be halved, and so on, the bits will become smaller and smaller, and can theoretically be made as small as we please. However small they may be, they can still be cut up and made smaller still. But they will always have *some* finite size, however small they may be. We never reach the infinitesimal in this way, and no finite number of divisions will bring us to points. Nevertheless there *are* points, only these are not to be reached by successive divisions. Here again, the philosophy of the infinite shows us how this is possible, and why points are not infinitesimal lengths.

As regards motion and change, we get similarly curious results. People used to think that when a thing changes, it must be in a state of change, and that when a thing moves, it is in a state of motion. This is now known to be a mistake. When a body moves, all that can be said is that it is in one place at one time and in another at another. We must not say that it will be in a neighbouring place at the next instant, since there is no next instant. Philosophers often tell us that when a body is in motion, it changes its position within the instant. To this view Zeno long ago made the fatal retort that every body always is where it is; but a retort so simple and brief was not of the kind to which philosophers are accustomed to give weight, and they have continued down to our own day to repeat the same phrases which roused the Eleatic's destructive ardour. It was only recently that it became possible to explain motion in detail in accordance with Zeno's platitude, and in opposition to the philosopher's paradox. We may now at last indulge the comfortable belief that a body in motion is just as truly where it is as a body at rest. Motion consists merely in the fact that bodies are sometimes in one place and sometimes in another, and that they are at intermediate places at intermediate times. Only those who have waded through the quagmire of philosophic speculation on this subject can realise what a liberation from antique prejudices is involved in this simple and straightforward commonplace.

The philosophy of the infinitesimal, as we have just seen, is mainly negative. People used to believe in it, and now they have found out their mistake. The philosophy of the infinite, on the other hand, is wholly positive. It was formerly supposed that infinite numbers, and the mathematical infinite generally, were self-contradictory. But as it was obvious that there were infinities—for example, the number of numbers—the contradictions of infinity seemed unavoidable, and philosophy seemed to have wandered into a "cul-de-sac." This difficulty led to Kant's antinomies, and hence, more or less indirectly, to much of Hegel's dialectic method.

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Almost all current philosophy is upset by the fact (of which very few philosophers are as yet aware) that all the ancient and respectable contradictions in the notion of the infinite have been once for all disposed of. The method by which this has been done is most interesting and instructive. In the first place, though people had talked glibly about infinity ever since the beginnings of Greek thought, nobody had ever thought of asking, What is infinity? If any philosopher had been asked for a definition of infinity, he might have produced some unintelligible rigmarole, but he would certainly not have been able to give a definition that had any meaning at all. Twenty years ago, roughly speaking, Dedekind and Cantor asked this question, and, what is more remarkable, they answered it. They found, that is to say, a perfectly precise definition of an infinite number or an infinite collection of things. This was the first and perhaps the greatest step. It then remained to examine the supposed contradictions in this notion. Here Cantor proceeded in the only proper way. He took pairs of contradictory propositions, in which both sides of the contradiction would be usually regarded as demonstrable, and he strictly examined the supposed proofs. He found that all proofs adverse to infinity involved a certain principle, at first sight obviously true, but destructive, in its consequences, of almost all mathematics. The proofs favourable to infinity, on the other hand, involved no principle that had evil consequences. It thus appeared that common sense had allowed itself to be taken in by a specious maxim, and that, when once this maxim was rejected, all went well.

The maxim in question is, that if one collection is part of another, the one which is a part has fewer terms than the one of which it is a part. This maxim is true of finite numbers. For example, Englishmen are only some among Europeans, and there are fewer Englishmen than Europeans. But when we come to infinite numbers, this is no longer true. This breakdown of the maxim gives us the precise definition of infinity. A collection of terms is infinite when it contains as parts other collections which have just as many terms as it has. If you can take away some of the terms of a collection, without diminishing the number of terms, then there are an infinite number of terms in the collection. For example, there are just as many even numbers as there are numbers altogether, since every number can be doubled. This may be seen by putting odd and even numbers together in one row, and even numbers alone in a row below:—

1, 2, 3, 4, 5, *ad infinitum.*
2, 4, 6, 8, 10, *ad infinitum.*

There are obviously just as many numbers in the row below as in the row above, because there is one below for each one above. This property, which was formerly thought to be a contradiction, is now transformed into a harmless definition of infinity, and shows, in the above case, that the number of finite numbers is infinite.

But the uninitiated may wonder how it is possible to deal with a number which cannot be counted. It is impossible to count up *all* the numbers, one by one, because, however many we may count, there are always more to follow. The fact is that counting is a very vulgar and elementary way of finding out how many terms there are in a collection. And in any case, counting gives us what mathematicians call the *ordinal* number of our terms; that is to say, it arranges our terms in an order or series, and its result tells us what type of series results from this arrangement. In other words, it is impossible to count things without counting some first and others afterwards, so that counting always has to do with order. Now when there are only a finite number of terms, we can count them in any order we like; but when there are an infinite number, what corresponds to counting will give us quite different results according to the way in which we carry out the operation. Thus the ordinal number, which results from what, in a general sense may be called counting, depends not only upon how many terms we have, but also (where the number of terms is infinite) upon the way in which the terms are arranged.

The fundamental infinite numbers are not ordinal, but are what is called *cardinal.* They are not obtained by putting our terms in order and counting them, but by a different method, which tells us, to begin with, whether two collections have the same number of terms, or, if not, which is the greater.[4] It does not tell us, in the way in which counting does, *what* number of terms a collection has; but if we define a number as the number of terms in such and such a collection, then this method enables us to discover whether some other collection that may be mentioned has more or fewer terms. An illustration will show how this is done. If there existed some country in which, for one reason or another, it was impossible to take a census, but in which it was known that every man had a wife and every woman a husband, then (provided polygamy was not a national institution) we should know, without counting, that there were exactly as many men as there were women in that country, neither more nor less. This method can be applied generally. If there is some relation which, like marriage, connects the things in one collection each with one of the things in another collection, and vice versa, then the two collections have the same number of terms. This was the way in which we found that there are as many even numbers as there are numbers. Every number can be doubled, and every even number can be halved, and each process gives just one number corresponding to the one that is doubled or halved. And

[4] [Note added in 1917.] Although some infinite numbers are greater than some others, it cannot be proved that of any two infinite numbers one must be the greater.

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in this way we can find any number of collections each of which has just as many terms as there are finite numbers. If every term of a collection can be hooked on to a number, and all the finite numbers are used once, and only once, in the process, then our collection must have just as many terms as there are finite numbers. This is the general method by which the numbers of infinite collections are defined.

But it must not be supposed that all infinite numbers are equal. On the contrary, there are infinitely more infinite numbers than finite ones. There are more ways of arranging the finite numbers in different types of series than there are finite numbers. There are probably more points in space and more moments in time than there are finite numbers. There are exactly as many fractions as whole numbers, although there are an infinite number of fractions between any two whole numbers. But there are more irrational numbers than there are whole numbers or fractions. There are probably exactly as many points in space as there are irrational numbers, and exactly as many points on a line a millionth of an inch long as in the whole of infinite space. There is a greatest of all infinite numbers, which is the number of things altogether, of every sort and kind. It is obvious that there cannot be a greater number than this, because, if everything has been taken, there is nothing left to add. Cantor has a proof that there is no greatest number, and if this proof were valid, the contradictions of infinity would reappear in a sublimated form. But in this one point, the master has been guilty of a very subtle fallacy, which I hope to explain in some future work.[5]

We can now understand why Zeno believed that Achilles cannot overtake the tortoise and why as a matter of fact he can overtake it. We shall see that all the people who disagreed with Zeno had no right to do so, because they all accepted premises from which his conclusion followed. The argument is this: Let Achilles and the tortoise start along a road at the same time, the tortoise (as is only fair) being allowed a handicap. Let Achilles go twice as fast as the tortoise, or ten times or a hundred times as fast. Then he will never reach the tortoise. For at every moment the tortoise is somewhere and Achilles is somewhere; and neither is ever twice in the same place while the race is going on. Thus the tortoise goes to just as many places as Achilles does, because each is in one place at one moment, and in another at any other moment. But if Achilles were to catch up with the tortoise, the places where the tortoise would have been would be only part of the places where Achilles would have been. Here, we must suppose, Zeno appealed to the maxim that the whole has

[5] Cantor was not guilty of a fallacy on this point. His proof that there is no greatest number is valid. The solution of the puzzle is complicated and depends upon the theory of types, which is explained in *Principia Mathematica*, Vol. I (Camb. Univ. Press, 1910). [Note added in 1917.]

more terms than the part.[6] Thus if Achilles were to overtake the tortoise, he would have been in more places than the tortoise; but we saw that he must, in any period, be in exactly as many places as the tortoise. Hence we infer that he can never catch the tortoise. This argument is strictly correct, if we allow the axiom that the whole has more terms than the part. As the conclusion is absurd, the axiom must be rejected, and then all goes well. But there is no good word to be said for the philosophers of the past two thousand years and more, who have all allowed the axiom and denied the conclusion.

The retention of this axiom leads to absolute contradictions, while its rejection leads only to oddities. Some of these oddities, it must be confessed, are very odd. One of them, which I call the paradox of Tristram Shandy, is the converse of the Achilles, and shows that the tortoise, if you give him time, will go just as far as Achilles. Tristram Shandy, as we know, employed two years in chronicling the first two days of his life, and lamented that, at this rate, material would accumulate faster than he could deal with it, so that, as years went by, he would be farther and farther from the end of his history. Now I maintain that, if he had lived for ever, and had not wearied of his task, then, even if his life had continued as eventfully as it began, no part of his biography would have remained unwritten. For consider: the hundredth day will be described in the hundredth year, the thousandth in the thousandth year, and so on. Whatever day we may choose as so far on that he cannot hope to reach it, that day will be described in the corresponding year. Thus any day that may be mentioned will be written up sooner or later, and therefore no part of the biography will remain permanently unwritten. This paradoxical but perfectly true proposition depends upon the fact that the number of days in all time is no greater than the number of years.

Thus on the subject of infinity it is impossible to avoid conclusions which at first sight appear paradoxical, and this is the reason why so many philosophers have supposed that there were inherent contradictions in the infinite. But a little practice enables one to grasp the true principles of Cantor's doctrine, and to acquire new and better instincts as to the true and the false. The oddities then become no odder than the people at the antipodes, who used to be thought impossible because they would find it so inconvenient to stand on their heads.

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Mathematical Creation

By HENRI POINCARÉ

THE genesis of mathematical creation is a problem which should intensely interest the psychologist. It is the activity in which the human mind seems to take least from the outside world, in which it acts or seems to act only of itself and on itself, so that in studying the procedure of geometric thought we may hope to reach what is most essential in man's mind.

This has long been appreciated, and some time back the journal called *L'enseignement mathématique*, edited by Laisant and Fehr, began an investigation of the mental habits and methods of work of different mathematicians. I had finished the main outlines of this article when the results of that inquiry were published, so I have hardly been able to utilize them and shall confine myself to saying that the majority of witnesses confirm my conclusions; I do not say all, for when the appeal is to universal suffrage unanimity is not to be hoped.

A first fact should surprise us, or rather would surprise us if we were not so used to it. How does it happen there are people who do not understand mathematics? If mathematics invokes only the rules of logic, such as are accepted by all normal minds; if its evidence is based on principles common to all men, and that none could deny without being mad, how does it come about that so many persons are here refractory?

That not every one can invent is nowise mysterious. That not every one can retain a demonstration once learned may also pass. But that not every one can understand mathematical reasoning when explained appears very surprising when we think of it. And yet those who can follow this reasoning only with difficulty are in the majority: that is undeniable, and will surely not be gainsaid by the experience of secondary-school teachers.

And further: how is error possible in mathematics? A sane mind should not be guilty of a logical fallacy, and yet there are very fine minds who do not trip in brief reasoning such as occurs in the ordinary doings of life, and who are incapable of following or repeating without error the mathematical demonstrations which are longer, but which after all are only an accumulation of brief reasonings wholly analogous to those they make so easily. Need we add that mathematicians themselves are not infallible?

The answer seems to me evident. Imagine a long series of syllogisms,

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and that the conclusions of the first serve as premises of the following: we shall be able to catch each of these syllogisms, and it is not in passing from premises to conclusion that we are in danger of deceiving ourselves. But between the moment in which we first meet a proposition as conclusion of one syllogism, and that in which we reencounter it as premise of another syllogism occasionally some time will elapse, several links of the chain will have unrolled; so it may happen that we have forgotten it, or worse, that we have forgotten its meaning. So it may happen that we replace it by a slightly different proposition, or that, while retaining the same enunciation, we attribute to it a slightly different meaning, and thus it is that we are exposed to error.

Often the mathematician uses a rule. Naturally he begins by demonstrating this rule; and at the time when this proof is fresh in his memory he understands perfectly its meaning and its bearing, and he is in no danger of changing it. But subsequently he trusts his memory and afterward only applies it in a mechanical way; and then if his memory fails him, he may apply it all wrong. Thus it is, to take a simple example, that we sometimes make slips in calculation because we have forgotten our multiplication table.

According to this, the special aptitude for mathematics would be due only to a very sure memory or to a prodigious force of attention. It would be a power like that of the whist-player who remembers the cards played; or, to go up a step, like that of the chess-player who can visualize a great number of combinations and hold them in his memory. Every good mathematician ought to be a good chess-player, and inversely; likewise he should be a good computer. Of course that sometimes happens; thus Gauss was at the same time a geometer of genius and a very precocious and accurate computer.

But there are exceptions; or rather I err; I can not call them exceptions without the exceptions being more than the rule. Gauss it is, on the contrary, who was an exception. As for myself, I must confess, I am absolutely incapable even of adding without mistakes. In the same way I should be but a poor chess-player; I would perceive that by a certain play I should expose myself to a certain danger; I would pass in review several other plays, rejecting them for other reasons, and then finally I should make the move first examined, having meantime forgotten the danger I had foreseen.

In a word, my memory is not bad, but it would be insufficient to make me a good chess-player. Why then does it not fail me in a difficult piece of mathematical reasoning where most chess-players would lose themselves? Evidently because it is guided by the general march of the reasoning. A mathematical demonstration is not a simple juxtaposition of syllogisms, it is syllogisms *placed in a certain order*, and the order in which

these elements are placed is much more important than the elements themselves. If I have the feeling, the intuition, so to speak, of this order, so as to perceive at a glance the reasoning as a whole, I need no longer fear lest I forget one of the elements, for each of them will take its allotted place in the array, and that without any effort of memory on my part.

It seems to me then, in repeating a reasoning learned, that I could have invented it. This is often only an illusion; but even then, even if I am not so gifted as to create it by myself, I myself re-invent it in so far as I repeat it.

We know that this feeling, this intuition of mathematical order, that makes us divine hidden harmonies and relations, can not be possessed by every one. Some will not have either this delicate feeling so difficult to define, or a strength of memory and attention beyond the ordinary, and then they will be absolutely incapable of understanding higher mathematics. Such are the majority. Others will have this feeling only in a slight degree, but they will be gifted with an uncommon memory and a great power of attention. They will learn by heart the details one after another; they can understand mathematics and sometimes make applications, but they cannot create. Others, finally, will possess in a less or greater degree the special intuition referred to, and then not only can they understand mathematics even if their memory is nothing extraordinary, but they may become creators and try to invent with more or less success according as this intuition is more or less developed in them.

In fact, what is mathematical creation? It does not consist in making new combinations with mathematical entities already known. Any one could do that, but the combinations so made would be infinite in number and most of them absolutely without interest. To create consists precisely in not making useless combinations and in making those which are useful and which are only a small minority. Invention is discernment, choice.

How to make this choice I have before explained; the mathematical facts worthy of being studied are those which, by their analogy with other facts, are capable of leading us to the knowledge of a mathematical law just as experimental facts lead us to the knowledge of a physical law. They are those which reveal to us unsuspected kinship between other facts, long known, but wrongly believed to be strangers to one another.

Among chosen combinations the most fertile will often be those formed of elements drawn from domains which are far apart. Not that I mean as sufficing for invention the bringing together of objects as disparate as possible; most combinations so formed would be entirely sterile. But certain among them, very rare, are the most fruitful of all.

To invent, I have said, is to choose; but the word is perhaps not wholly exact. It makes one think of a purchaser before whom are displayed a large number of samples, and who examines them, one after the other, to

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make a choice. Here the samples would be so numerous that a whole lifetime would not suffice to examine them. This is not the actual state of things. The sterile combinations do not even present themselves to the mind of the inventor. Never in the field of his consciousness do combinations appear that are not really useful, except some that he rejects but which have to some extent the characteristics of useful combinations. All goes on as if the inventor were an examiner for the second degree who would only have to question the candidates who had passed a previous examination.

But what I have hitherto said is what may be observed or inferred in reading the writings of the geometers, reading reflectively.

It is time to penetrate deeper and to see what goes on in the very soul of the mathematician. For this, I believe, I can do best by recalling memories of my own. But I shall limit myself to telling how I wrote my first memoir on Fuchsian functions. I beg the reader's pardon; I am about to use some technical expressions, but they need not frighten him, for he is not obliged to understand them. I shall say, for example, that I have found the demonstration of such a theorem under such circumstances. This theorem will have a barbarous name, unfamiliar to many, but that is unimportant; what is of interest for the psychologist is not the theorem but the circumstances.

For fifteen days I strove to prove that there could not be any functions like those I have since called Fuchsian functions. I was then very ignorant; every day I seated myself at my work table, stayed an hour or two, tried a great number of combinations and reached no results. One evening, contrary to my custom, I drank black coffee and could not sleep. Ideas rose in crowds; I felt them collide until pairs interlocked, so to speak, making a stable combination. By the next morning I had established the existence of a class of Fuchsian functions, those which come from the hypergeometric series; I had only to write out the results, which took but a few hours.

Then I wanted to represent these functions by the quotient of two series; this idea was perfectly conscious and deliberate, the analogy with elliptic functions guided me. I asked myself what properties these series must have if they existed, and I succeeded without difficulty in forming the series I have called theta-Fuchsian.

Just at this time I left Caen, where I was then living, to go on a geological excursion under the auspices of the school of mines. The changes of travel made me forget my mathematical work. Having reached Coutances, we entered an omnibus to go some place or other. At the moment when I put my foot on the step the idea came to me, without anything in my former thoughts seeming to have paved the way for it, that the transformations I had used to define the Fuchsian functions were identical with

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those of non-Euclidean geometry. I did not verify the idea; I should not have had time, as, upon taking my seat in the omnibus, I went on with a conversation already commenced, but I felt a perfect certainty. On my return to Caen, for conscience' sake I verified the result at my leisure.

Then I turned my attention to the study of some arithmetic questions apparently without much success and without a suspicion of any connection with my preceding researches. Disgusted with my failure, I went to spend a few days at the seaside, and thought of something else. One morning, walking on the bluff, the idea came to me, with just the same characteristics of brevity, suddenness and immediate certainty, that the arithmetic transformations of indeterminate ternary quadratic forms were identical with those of non-Euclidean geometry.

Returned to Caen, I meditated on this result and deduced the consequences. The example of quadratic forms showed me that there were Fuchsian groups other than those corresponding to the hypergeometric series; I saw that I could apply to them the theory of theta-Fuchsian series and that consequently there existed Fuchsian functions other than those from the hypergeometric series, the ones I then knew. Naturally I set myself to form all these functions. I made a systematic attack upon them and carried all the outworks, one after another. There was one however that still held out, whose fall would involve that of the whole place. But all my efforts only served at first the better to show me the difficulty, which indeed was something. All this work was perfectly conscious.

Thereupon I left for Mont-Valérien, where I was to go through my military service; so I was very differently occupied. One day, going along the street, the solution of the difficulty which had stopped me suddenly appeared to me. I did not try to go deep into it immediately, and only after my service did I again take up the question. I had all the elements and had only to arrange them and put them together. So I wrote out my final memoir at a single stroke and without difficulty.

I shall limit myself to this single example; it is useless to multiply them. In regard to my other researches I would have to say analogous things, and the observations of other mathematicians given in *L'enseignement mathématique* would only confirm them.

Most striking at first is this appearance of sudden illumination, a manifest sign of long, unconscious prior work. The rôle of this unconscious work in mathematical invention appears to me incontestable, and traces of it would be found in other cases where it is less evident. Often when one works at a hard question, nothing good is accomplished at the first attack. Then one takes a rest, longer or shorter, and sits down anew to the work. During the first half-hour, as before, nothing is found, and then all of a sudden the decisive idea presents itself to the mind. It might be said that the conscious work has been more fruitful because it has been interrupted and the rest has given back to the mind its force and freshness. But it is more probable that this rest has been filled out with unconscious work and that the result of this work has afterwards revealed itself to the geometer just as in the cases I have cited; only the revelation, instead of coming during a walk or a journey, has happened during a period of conscious work, but independently of this work which plays at most a rôle of excitant, as if it were the goad stimulating the results already reached during rest, but remaining unconscious, to assume the conscious form.

There is another remark to be made about the conditions of this unconscious work: it is possible, and of a certainty it is only fruitful, if it is on the one hand preceded and on the other hand followed by a period of conscious work. These sudden inspirations (and the examples already cited sufficiently prove this) never happen except after some days of voluntary effort which has appeared absolutely fruitless and whence nothing good seems to have come, where the way taken seems totally astray. These efforts then have not been as sterile as one thinks; they have set agoing the unconscious machine and without them it would not have moved and would have produced nothing.

The need for the second period of conscious work, after the inspiration, is still easier to understand. It is necessary to put in shape the results of this inspiration, to deduce from them the immediate consequences, to arrange them, to word the demonstrations, but above all is verification necessary. I have spoken of the feeling of absolute certitude accompanying the inspiration; in the cases cited this feeling was no deceiver, nor is it usually. But do not think this a rule without exception; often this feeling deceives us without being any the less vivid, and we only find it out when we seek to put on foot the demonstration. I have especially noticed this fact in regard to ideas coming to me in the morning or evening in bed while in a semi-hypnagogic state.

Such are the realities; now for the thoughts they force upon us. The unconscious, or, as we say, the subliminal self plays an important rôle in mathematical creation; this follows from what we have said. But usually the subliminal self is considered as purely automatic. Now we have seen that mathematical work is not simply mechanical, that it could not be done by a machine, however perfect. It is not merely a question of applying rules, of making the most combinations possible according to certain fixed laws. The combinations so obtained would be exceedingly numerous, useless and cumbersome. The true work of the inventor consists in choosing among these combinations so as to eliminate the useless ones or rather to avoid the trouble of making them, and the rules which must guide this choice are extremely fine and delicate. It is almost impossible to state them precisely; they are felt rather than formulated. Under these conditions, how imagine a sieve capable of applying them mechanically?

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A first hypothesis now presents itself: the subliminal self is in no way inferior to the conscious self; it is not purely automatic; it is capable of discernment; it has tact, delicacy; it knows how to choose, to divine. What do I say? It knows better how to divine than the conscious self, since it succeeds where that has failed. In a word, is not the subliminal self superior to the conscious self?

Hans Hahn: *The crisis in intuition*

Hans Hahn lectured at Vienna during the 1920s on *The crisis in intuition.* We present here some extracts from Hahn's lectures:-

The crisis in intuition

Immanuel Kant, in his *Critique of Pure Reason,* [has asserted that] ... we conduct ourselves passively when we receive impressions through intuition and actively when we deal with them in our thought. Furthermore, according to Kant, we must distinguish between two ingredients of intuition. One ... arises from experience ... such as colours, sounds, smells, hardness, softness, roughness, etc. The other is a pure a priori part independent of all experience ..: [Kant believed that] geometry, as it has been taught since ancient times, deals with the properties of the space that is fully and exactly presented to us by pure intuition ....

However plausible these ideas may at first seem, and however well they corresponded to the state of science in Kant's day, their foundations have been shaken by the course that science has taken since then ....

[These quotes] narrow the subject to geometry and intuition, and attempt to show how it came about that, even in the branch of mathematics which would seem to be its original domain, intuition gradually fell into disrepute and at last was completely banished ....

One of the outstanding events in this development was the discovery [by Weierstrass of] curves that possess no tangent at any point. [That is,] ... it is possible to imagine a point moving in such a manner that at no instant does it have a definite velocity. [This] directly affects the foundations of differential calculus as developed by Newton (who started with the concept of velocity) and Leibniz (who started the so-called tangent problem) .... The standard curves that have been studied since early times: circles, ellipses, hyperbolas, parabolas, cycloids, etc. [have tangents everywhere. However,] the graph of the function $t \cos(1/t)$ demonstrates that a curve does not have to have a tangent at every point. It used to be thought that intuition forced us to acknowledge that such a deficiency could occur only at isolated and exceptional points of a curve [and] that a curve must possess an exact slope, or tangent, at an overwhelming majority of points [The Weierstrass function goes beyond that. By replacing lines with saw-tooth curves, one obtains a simplified variant, the Takagi function ...] Its character entirely eludes intuition: indeed, after a few repetitions of the segmenting process, the evolving figure has grown so intricate that intuition can scarcely follow, and it forsakes us completely as regards the curve that is approached as a limit. The fact is that only logical analysis can pursue this strange object to its final form. Thus, had we relied on intuition in this instance, we would have remained in error, for intuition seems to force the conclusion that there cannot be curves lacking a tangent at any point .... [To avoid such advanced] branches of mathematics, I propose to examine an occurrence of failure of intuition at the very threshold of geometry. Everyone believes that ... curves are geometric figures generated by the motion of a point. But ... Peano ... proved that the geometric figures that can be generated by a moving point also include entire plane surfaces. For instance, it is possible to imagine a point moving in such a way that in a finite time it will pass through all the points of a square and yet no one would consider the entire area of a square as simply a curve .... This motion cannot possibly be grasped by intuition; it can only be understood by logical analysis.

[For] a second example of the undependability of intuition even as regards very elementary geometrical questions, think of a map showing three countries.

Intuition seems to indicate that corners at which all three countries come together ... can occur only at isolated points, and that at the great majority of boundary points on the map only two countries will be in contact. Yet Brouwer showed how a map can be divided into three countries in such a way that at every boundary point all three countries will touch one another ....

Intuition cannot comprehend this pattern, although logical analysis requires us to accept it. Once more intuition has led us astray. Intuition seems to indicate that it is impossible for a curve to be made up of nothing but end points or of branch points. This intuitive conviction as regards branch points was refuted [when] Sierpinski proved that there are curves *all of whose points are branch points ....*

Because intuition turned out to be deceptive in so many instances, and because propositions that had been accounted true by intuition were repeatedly proved false by logic, mathematicians became more and more skeptical of the validity of intuition. They learned that it is unsafe to accept any mathematical proposition, much less to base any mathematical discipline on intuitive convictions. Thus, a demand arose for the expulsion of intuition from mathematical reasoning, and for the complete formalization of mathematics. That is to say, every new mathematical concept was to be introduced through a purely logical definition; every mathematical proof was to be carried through strictly by logical means. The task of completely formalizing mathematics, of reducing it entirely to logic, was arduous and difficult; it meant nothing less than a reform in root and branch ....

Let us now summarize. Again and again we have found that, even in simple and elementary geometric questions, intuition is a wholly unreliable guide. It is impossible to permit so unreliable an aid to serve as the starting point or basis of a mathematical discipline ...

But what are we to say to the often heard objection that only conventional geometry is usable, for it is the only one that satisfies intuition? My first comment on this score ... is that *every* geometry ... is a logical construct. Traditional physics is responsible for the fact that until recently the logical construction of three-dimensional Euclidean, Archimedean space has been used exclusively for the ordering of our experience. For several centuries, almost up to the present day, it served this purpose admirably; thus we grew used to operating with it. This habituation to the use of ordinary geometry for the ordering of our experience explains why we regard this geometry as intuitive, and every departure from it unintuitive, contrary to intuition, and intuitively impossible. But as we have seen, such intuitional impossibilities, also occur in ordinary geometry. They appear as soon as we no longer restrict ourselves to the geometrical entities with which we have long been familiar, but instead reflect upon objects that we had not thought about before ....

The theory that the earth is a sphere was also once an affront to intuition. However, we have got used to the idea, and today it no longer occurs to anyone to pronounce it impossible because it conflicts with intuition.

If the use of [new] geometries for the ordering of our experience continues to prove itself so that we become more and more accustomed to dealing with these logical constructs; if they penetrate into the curriculum of the schools, if we, so to speak, learn them at our mother's knee, as we now learn three-dimensional Euclidean geometry, then nobody will think of saying that these geometries are contrary to intuition. They will be considered as deserving of intuitive status as three-dimensional Euclidean geometry is today. For it is not true, as Kant urged, that intuition is a pure *a priori* means of knowledge, but rather that it is force of habit rooted in psychological inertia.

1 Paradox Lost and Paradox Regained

By EDWARD KASNER
and JAMES R. NEWMAN

How quaint the ways of paradox—
At common sense she gaily mocks.
—W. S. GILBERT

PERHAPS the greatest paradox of all is that there are paradoxes in mathematics. We are not surprised to discover inconsistencies in the experimental sciences, which periodically undergo such revolutionary changes that although only a short time ago we believed ourselves descended from the gods, we now visit the zoo with the same friendly interest with which we call on distant relatives. Similarly, the fundamental and age-old distinction between matter and energy is vanishing, while relativity physics is shattering our basic concepts of time and space. Indeed, the testament of science is so continuously in a flux that the heresy of yesterday is the gospel of today and the fundamentalism of tomorrow. Paraphrasing Hamlet—what was once a paradox is one no longer, but may again become one. Yet, because mathematics builds on the old but does not discard it, because it is the most conservative of the sciences, because its theorems are deduced from postulates by the methods of logic, in spite of its having undergone revolutionary changes we do not suspect it of being a discipline capable of engendering paradoxes.

Nevertheless, there are three distinct types of paradoxes which do arise in mathematics. There are contradictory and absurd propositions, which arise from fallacious reasoning. There are theorems which seem strange and incredible, but which, because they are logically unassailable, must be accepted even though they transcend intuition and imagination. The third and most important class consists of those logical paradoxes which arise in connection with the theory of aggregates, and which have resulted in a re-examination of the foundations of mathematics. These logical para-

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doxes have created confusion and consternation among logicians and mathematicians and have raised problems concerning the nature of mathematics and logic which have not yet found a satisfactory solution.

(some pages have been skipped.)

LOGICAL PARADOXES

Like folk tales and legends, the logical paradoxes had their forerunners in ancient times. Having occupied themselves with philosophy and with the foundations of logic, the Greeks formulated some of the logical conundrums which, in recent times, have returned to plague mathematicians and philosophers. The Sophists made a specialty of posers to bewilder and confuse their opponents in debate, but most of them rested on sloppy thinking and dialectical tricks. Aristotle demolished them when he laid down the foundations of classical logic—a science which has outworn and outlasted all the philosophical systems of antiquity, and which, for the most part, is perfectly valid today.

But there were troublesome riddles that stubbornly resisted unravelling.[12] Most of them are caused by what is known as "the vicious circle fallacy," which is "due to neglecting the fundamental principle that what involves the whole of a given totality cannot itself be a member of the totality." [13] Simple instances of this are those pontifical phrases, familiar to everyone, which seem to have a great deal of meaning, but actually have none, such as "never say never," or "every rule has exceptions," or, "every generality is false." We shall consider a few of the more advanced logical paradoxes involving the same basic fallacy, and then discuss their importance from the mathematician's point of view.

(A) Poaching on the hunting preserves of a powerful prince was punishable by death, but the prince further decreed that anyone caught poaching was to be given the privilege of deciding whether he should be hanged or beheaded. The culprit was permitted to make a statement—if it were false, he was to be hanged; if it were true, he was to be beheaded. One logical rogue availed himself of this dubious prerogative—to be hanged if he didn't and to be beheaded if he did—by stating: "I shall be hanged." Here was a dilemma not anticipated. For, as the poacher put it, "If you now hang me, you break the laws made by the prince, for my statement is true, and I ought to be beheaded; but if you behead me, you are also breaking the laws, for then what I said was false and I should, therefore, be hanged." As in Frank Stockton's story of the lady and the tiger, the ending is up to you. However, the poacher probably fared no worse at the hands of the executioner than he would have at the hands of a philosopher, for until this century philosophers had little time to waste on such childish riddles—especially those they could not solve.

(B) The village barber shaves everyone in the village who does not shave himself. But this principle soon involves him in a dialectical plight analogous to that of the executioner. Shall he shave himself? If he does, then he is shaving someone who shaves himself and breaks his own rule. If he does not, besides remaining unshaven, he also breaks his rule by failing to shave a person in the village who does not shave himself.

(C) Consider the fact that every integer may be expressed in the English language without the use of symbols. Thus, (a) 1400 may be written

[12] For instance, the riddle of the Epimenides concerning the Cretan who says that all Cretans are liars.

[13] Ramsay, Frank Plumpton. Articles on "Mathematics," and "Logic," *Encyclopaedia Britannica*, 13th edition.

as one thousand, four hundred, or (b) 1769823 as one million, seven hundred and sixty-nine thousand, eight hundred and twenty-three. It is evident that certain numbers require more syllables than others; in general, the larger the integer, the more syllables needed to express it. Thus, (a) requires 6 syllables, and (b) 21. Now, it may be established that certain numbers will require 19 syllables or less, while others will require more than 19 syllables. Furthermore, it is not difficult to show that among those integers requiring exactly 19 syllables to be expressed in the English language, there must be a smallest one. Now, "it is easy to see" [14] that "*The least integer not nameable in fewer than nineteen syllables*" is a phrase which must denote the specific number, 111777. But the italicized expression above is itself an unambiguous means of denoting the smallest integer expressible in nineteen syllables in the English language. Yet, the italicized statement has only eighteen syllables! Thus, we have a contradiction, for the least integer expressible in nineteen syllables can be expressed in eighteen syllables.

(D) The simplest form of the logical paradox which arises from the indiscriminate use of the word *all* may be seen in Figure 9.

What is to be said about the statement numbered 3? 1 and 2 are false, but 3 is both a wolf dressed like a sheep and a sheep dressed like a wolf. It is neither the one thing nor the other: It is neither false nor true.

An elaboration appears in the famous paradox of Russell about the

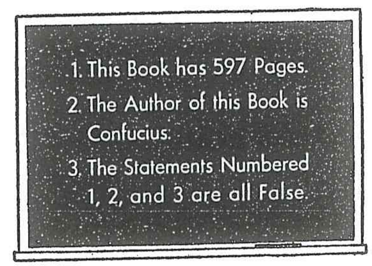

FIGURE 9

class of all classes not members of themselves. The thread of the argument is somewhat elusive and will repay careful attention:

(E) Using the word class in the customary sense, we can say that there are classes made up of tables, books, peoples, numbers, functions, ideas, etc. The class, for instance, of all the Presidents of the United

[14] This expression may, perhaps, be taken in the sense in which Laplace employed it. When he wrote his monumental *Mécanique Céleste*, he made abundant use of the expression, "It is easy to see" often prefixing it to a mathematical formula which he had arrived at only after months of labor. The result was that scientists who read his work almost invariably recognized the expression as a danger signal that there was very rough going ahead.

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States has for its members every person, living or dead, who was ever President of the United States. Everything in the world other than a person who was or is a President of the United States, including the *concept* of the class itself is *not* a member of this class. This then, is an example of a class which is *not* a member of itself. Likewise, the class of all members of the Gestapo, or German secret police, which contains some, but not all, of the scoundrels in Germany; or the class of all geometric figures in a plane bounded by straight lines; or the class of all integers from one to four thousand inclusive, have for members, the things described, but the classes are not members of themselves.

Now, if we consider a class as a *concept*, then the *class* of *all concepts* in the world is itself a concept, and thus is a class which is a member of itself. Again, the class of all ideas brought to the attention of the reader in this book is a class which contains itself as a member, since in mentioning this class, it is an idea which we bring to the attention of the reader. Bearing this distinction in mind, we may divide all classes into two types. Those which are members of themselves and those which are *not* members of themselves. Indeed, we may form a class which is composed of *all those classes which* are *not* members of themselves (note the dangerous use of the word "all"). The question is presented: Is this class (composed of those classes which are not members of themselves) a member of itself, or not? Either an affirmative or a negative answer involves us in a hopeless contradiction. If the class in question *is* a member of itself, it ought not be by definition, for it should contain only those classes which are not members of themselves. But if it is *not* a member of itself, it ought to be a member of itself, for the same reason.

It cannot be too strongly emphasized that the logical paradoxes are not idle or foolish tricks. They were not included in this volume to make the reader laugh, unless it be at the limitations of logic. The paradoxes are like the fables of La Fontaine which were dressed up to look like innocent stories about fox and grapes, pebbles and frogs. For just as all ethical and moral concepts were skillfully woven into their fabric, so all of logic and mathematics, of philosophy and speculative thought, is interwoven with the fate of these little jokes.

Modern mathematics, in attempting to avoid the paradoxes of the theory of aggregates, was squarely faced with the alternatives of adopting annihilating skepticism in regard to all mathematical reasoning, or of reconsidering and reconstructing the foundations of mathematics as well as logic. It should be clear that if paradoxes can arise from apparently legitimate reasoning about the theory of aggregates, they may arise *anywhere* in mathematics. Thus, even if mathematics could be reduced to logic, as Frege and Russell had hoped, what purpose would be served if logic itself were insecure? In proposing their "Theory of Types" Whitehead and

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Russell, in the *Principia Mathematica*, succeeded in avoiding the contradictions by a formal device. Propositions which were *grammatically* correct but contradictory, were branded as meaningless. Furthermore, a principle was formulated which specifically states what form a proposition must take to be meaningful; but this solved only half the difficulty, for although the contradictions could be recognized, the arguments leading to the contradictions could not be invalidated without affecting certain accepted portions of mathematics. To overcome this difficulty, Whitehead and Russell postulated the *axiom of reducibility* which, however, is too technical to be considered here. But the fact remains that the axiom is not acceptable to the great majority of mathematicians and that the logical paradoxes, having divided mathematicians into factions unalterably opposed to each other, have still to be disposed of.[15]

* * * * *

It has been emphasized throughout that the mathematician strives always to put his theorems in the most general form. In this respect, the aims of the mathematician and the logician are identical—to formulate propositions and theorems of the form: if A is true, B is true, where A and B embrace much more than merely cabbages and kings. But if this is a high aim, it is also dangerous, in the same way that the concept of the infinite is dangerous. When the mathematician says that such and such a proposition is true of one thing, it may be interesting, and it is surely safe. But when he tries to extend his proposition to *everything*, though it is much more interesting, it is also much more dangerous. In the transition from *one* to *all*, from the specific to the general, mathematics has made its greatest progress, and suffered its most serious setbacks, of which the logical paradoxes constitute the most important part. For, if mathematics is to advance securely and confidently it must first set its affairs in order at home.

[15] As was pointed out in discussing the googol,* there are the followers of Russell who are satisfied with the theory of types and the axiom of reducibility; there are the Intuitionists, led by Brouwer and Weyl, who reject the axiom and whose skepticism about the infinite in mathematics has carried them to the point where they would reject large portions of modern mathematics as meaningless, because they are interwoven with the infinite; and there are the Formalists, led by Hilbert, who, while opposed to the beliefs of the Intuitionists, differ considerably from Russell and the Logistic school. It is Hilbert who considers mathematics a meaningless game, comparable to chess, and he has created a subject of metamathematics which has for its program the discussion of this meaningless game and its axioms.

* [For the meaning of "googol" see selection by Kasner and Newman, "New Names for Old," p. 2007.]

THE FOLLOWING OPTICAL ILLUSIONS, WHILE NOT PROPERLY PART OF A BOOK ON MATHEMATICS, MAY BE OF SOME INTEREST—AT LEAST TO THE IMAGINATION

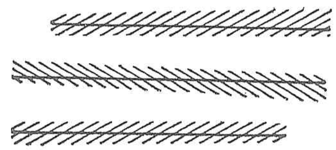

PLATE I—Are the three horizontal lines parallel?

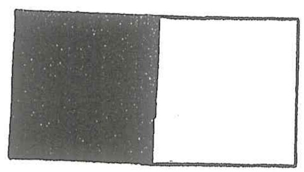

PLATE II—The white square is of course larger than the black. Or is it smaller?

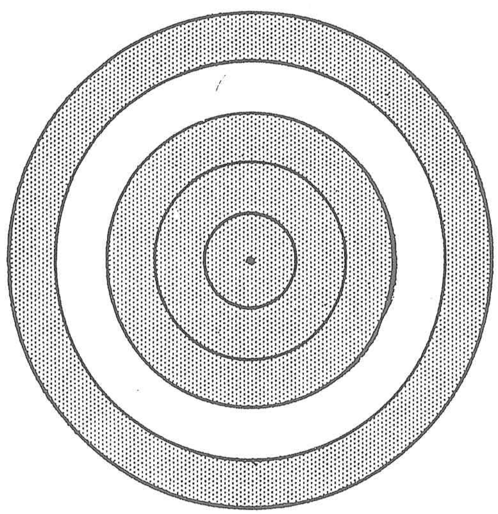

PLATE III—The two shaded regions have the same area.

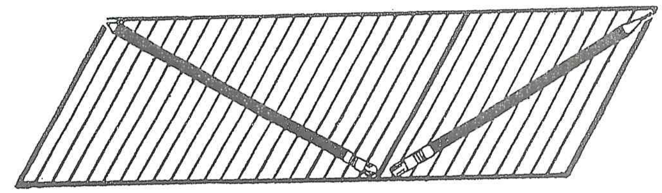

PLATE IV—Which of the two pencils is longer? Measure them and find out.

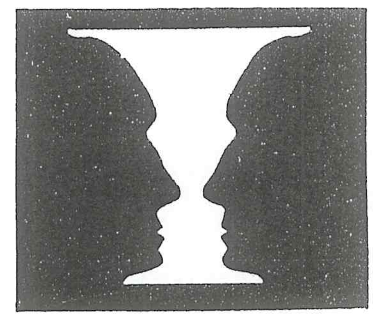

PLATE V—What do you see? Now look again.

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The Western Heritage from the earliest times to the Present. By Stewart Easton.

Daily life in Athens

The whole social life of Athens, and indeed that of all other Greek cities, reflects, above all, the extraordinary poverty of material resources, which was not only accepted philosophically by the Greeks but regarded as the natural and even desirable order of things. The ordinary man remained a frugal liver, both in imperial times and in the fourth century. Even what he considered luxuries would be to the imperial Roman very little indeed. All Greeks wore clothes of the utmost simplicity at all times, an undergarment fastened with a safety pin, and an outer garment draped about their person. The same garment served as a blanket. Beds were usually planks, without springs. The average house, unlike the temples, was made of sun-dried brick, and houses were built close together. The walls were not decorated, the furniture was crude and utilitarian.

The reason for this utter lack of luxury in the private homes of the Athenians is simple enough. The Greek lived primarily in the open air. More hours of the day were spent in the gymnasium, the agora, or the streets than in his house. When it was dark he went to bed, and at dawn he usually rose and went into the street, without breakfast. We hear nothing from any source of any great mansions of the Roman type in classical times, nor of palatial private gardens and pleasure grounds. Rich men contributed their wealth to the polis, and did not use it so much for their own pleasure; but even their riches were small enough by Roman or Cretan standards. There were no gargantuan feasts; food was scarce and lacked variety. Meat was rarely eaten.

The truth seems to be, hard as it may be for us to believe, that the Greek really did not care for luxury, or not enough to give up his leisure to gain it; and it was frowned upon by public opinion. A contrast sometimes made between Athenian luxury and Spartan simplicity is extremely relative. Both lived simply; but the Spartan cultivated simplicity, wearing only one garment in winter and going barefoot, whereas the Athenian had sandals. The Athenian was able to decorate his city superbly because he cared for it rather than for his home; and to the service of his gods and his city he devoted all his unparalleled artistic talents.

The kind of freedom that resulted from this doing without is one that is unique in history, and can never be repeated. But if one delights in free talk, assemblies, festivals, plays, the development of the mind and the body, self-government, and civic glory, the logical thing to do is to avoid cluttering oneself up with possessions useless to this kind of life.

In the rest of this chapter we shall see how they used this freedom.

▸ The searching mind of the Greek

GENERAL CHARACTERISTICS OF GREEK THOUGHT

All knowledge, said the Greeks, begins in wonder—wonder about the world, and wonder about man. The Hebrews asked only one question about man: his relation to his God. The Greeks asked not only this question but all other questions. They were the greatest people for questioning that the world has yet seen, or at all events until our own time.

The Greeks wondered about the physical world. What was the underlying stable substratum in a world where everything appeared to be in flux—was it water, air, fire, or atoms? Clearly everything changed in appearance; but they did not doubt that this change was only an apparent change. Underneath was a unity.

They wondered about man—his nature, the seen body, and the unseen soul that gave life to it. They assumed the existence of the soul, but they tried to find the relationship between soul and body. How does man acquire knowledge? What is the nature of the mind that knows it? What are the laws of thinking? How does one idea connect with another? What is an idea? What are the activities proper to man? What is morality?

In all these questions except the last, the Greeks were pioneers in human thinking; and even in the last they were different from the Hebrews in that at least the later Greeks accepted nothing, not even the gods, as final arbiters.

Constructive philosophy—Idealism of Plato

The heart of Plato's teaching stems from the original conception of Socrates that the human being can know the good; and that, knowing it, he can do it. What Plato seeks to discover is *how* he can know it, and *what* it is exactly that he knows. And by using the dialogue form he shows us the whole process by which he arrived at his conclusions; hence the endless stimulation that Plato has afforded to all subsequent mankind. All that we must do is hitch up to his thought at one place, and either follow him to the same conclusions, or, by casting aside some of his thoughts as based on assumptions which we will not accept, proceed to arrive at different conclusions.

Assuming, then, that man can know the good, with what faculty does he know it, and what is the object of this faculty? To this Plato answers that man is possessed of the power of thinking (*Nous*), and that this spiritual element in man can recognize the spiritual element akin to it—the Idea. And this Idea is not in the physical world, but in the spiritual world, forever hidden from every faculty in man save the Nous. Following this thought further, he concludes that everything we see in front of us is a *particular,* a single example of something, the Idea or archetype of which is really spiritual, and not to be found on earth. We see, for instance, a single plant; but the Idea of the plant is in the spiritual world. From this it is but a short step to the value judgment that the earthly example is necessarily an inferior copy of the ideal plant—that the spiritual reality is more beautiful, more worthy of contemplation than anything on earth.

The next step is to consider how we can recognize this earthly copy as indeed a copy of an Idea. And to this Plato's answer is that the soul, with its active faculty, the Nous, existed before incarnation on earth in a human body. Before it descended to earth it glimpsed these Ideas, which were implanted forever in the soul. Thus knowledge of the universal behind the particular appearance on earth is simply *recognition.* This, it will be seen, completely accounts for man's possession of innate knowledge, which Socrates had shown man did possess.

It is clear that this "idealist" philosophy gives an enormous scope to the philosopher. He is not compelled to examine the phenomena in front of him but may reason a priori; indeed, since it is only human thinking that can perceive the Ideas, there is no other method of reasoning than a priori. Thus by reasoning, the moral and political philosopher must try to discover for himself the ideal good, and not the practicable good.

The *Republic* is the Platonic masterpiece of this kind of reasoning. But by this it should not be thought that Plato had no practical ends in view. He tells us specifically that he has. No political state of which he has knowledge has been *thought* out; all are defective. But in his view these defects need not be inevitable. For if men know the good they will not deliberately prefer the evil unless they have been warped beyond cure. Since "virtue" may be taught, men can be educated to admire the best, and not choose a second-best polity to live in.

His method, then, is to discover what is the bond which holds society together (justice), and then try to arrive at a definition of justice. He comes to the conclusion that justice in the citizen and in the state is identical, and that if each man is given a position in the social order which enables him to do that for which he is best fitted, and he performs this task properly, then the ends of both the citizen and the state will be fully served, and the society will be a just one. Plato then proceeds to inquire into how human potentialities can best be realized in a social framework, and what will be the nature of the social institutions required.

Given his premises, the whole work, built up on these lines, is logically impeccable. Its value in all ages has been its suggestiveness, and the joy of following the thought of a truly creative mind, willing to pursue the argument wherever it will lead, without deference to conventional Greek notions, as, for instance, on the inequality of women. It is not native conservatism or a preference for oligarchy—though these may have been present, they are irrelevant—that forces him to the conclusion that the enlightened despotism of a board of professional guardians (philosopher kings and queens) is the only possible "best" government. These alone have been able to discover the good, and they must be dedicated utterly to its pursuit, without the warping of judgment which would arise from the possession of either material goods or family. With such a body of truly scientific professionals there would be no need for laws or for the exercise of power; for at all grades in the society each man would have received the education, and hold the position, for which he was best fitted.

It has often been pointed out, justly, that Plato makes a number of assumptions which are extremely questionable—for instance, that public and private virtue are identical, and that a state made up of good individuals will be able to function harmoniously as a state. But it will usually be found that these assumptions are the result of his fundamental belief that no one, knowing the good, would deliberately choose to do evil. If the state is a just one, its duties will be just and good; the individual, if he is good, will desire to do this duty. Duty and inclination must coincide. If they do not, then either the state needs to be corrected or the individual needs to be improved—by development and adjustment, not by repression and force.

Plato may also be accused of neglecting the psychology of man, as it *must* have been known to him from experience. What was the use of theorizing about an ideal state when he knew of its impossibility in real life? Again the answer must be that by showing men the ideal good which was, for him, having regard to his assumptions, not impossible of realization but only extremely difficult, he was pointing out a direction for the aspirations and endeavors of man. And that it was not his last thought on the subject is shown by his later works, the *Statesman* and *Laws*, in which he outlines the "second-best state," the state ruled by laws, laws which are directed to the ethical improvement of man, but cannot be as scientifically impeccable as the personal guidance of the philosopher kings. And elsewhere he shows that he is not unaware of human psychology. He recognizes the irrational part of man, but does not consider it incurable. The desires are controlled by reason, which, in the light of its knowledge of the good, will give man the power of evaluating his desires at their true worth.

As with the state, so with man. The harmonious functioning of all the parts that go to make up the full man, this is self-realization under the guiding power of the Nous. It is a psychology the truth of which would be vehemently denied by both Christians and Freudians, who both deny the power of the mind to control the will unaided. Perhaps to these the psychology of Plato would seem naïve; but it was the fullest and most complete expression of the Greek ideal of harmony and *sophrosyne*, and of the Greek belief in the efficacy of human thinking. If it is a glorification of the one specifically human power, this to the Greeks would have been a recommendation. Oedipus to the Greeks was not a complex but a human being, proud and erring but undefeated; and they were glad to be considered of his company.

Philosophy becomes science—Aristotle

The universality of his genius—Aristotle was the son of a Chalcidian physician in the service of Philip of Macedon. He studied at the Academy of Plato and was unquestionably his most brilliant pupil. He was tutor of Alexander, son of Philip, for several years, returning to Athens and opening a school himself (the Lyceum), where he taught for twelve years. Forced into exile on the death of Alexander, he died a year later in 322 B.C. at the age of sixty-two.

Thus Aristotle stands at the end of the Classical Age of Greece before the great emigration to Asia that followed the conquests of Alexander; and in a very real sense he completed it. Though he left one or two things undone which were repaired by Theophrastus his pupil and successor (for instance, a work on plants and another on human character) and he contributed nothing to Greek mathematics, which followed an independent course, in other respects he took all the varied speculations of his Greek predecessors, brilliant and disorganized as they were, and by the giant force of his capacity for system, order, and classification, discharged them from his hands as sciences, a body of work that could be communicated to others in comprehensible form. Once he had laid down the principles of scientific inquiry, the work would not have to be done again. He was the first true scientist in the history of mankind; and few who have really studied his work would dispute his title to be the greatest the world has yet known. And now that we have passed beyond recovery into a world of specialists, there never will be anyone again who will be able to lay claim to the universality of his learning. Any one of half a dozen of his mental achievements would have entitled him to an undying fame. The sum total is almost beyond belief.

The laws of thinking—Logic—If this seem excessive praise, let us consider for a moment a few of Aristotle's achievements. Basing his observations upon Plato's theory of ideas, he formulated the laws of thinking, the relation between the universal and the particular, the formal procedure required for arriving at conclusions and correct reasoning, giving in passing a different solution to the problem of the origin of the universal. Disturbed by the way in which objects are described without including all their features, he formulated a method for describing them inclusively (the "categories of being"). Stimulated perhaps by Socrates' remark that he himself knew that his will prevented him from going into exile and not "his bones and sinews," as Anaxagoras would have claimed, he formulated a system for dealing accurately with causation and had to invent a new vocabulary for the purpose. Faced with a mass of biological data, he evolved a system of classification into genus and species which has been followed with modifications ever since.

The foundation for classification of phenomena—Genus and species—Aristotle is usually praised in these days rather patronizingly for his excellent and careful observation and description of the animal world, and his early recognition of facts which modern science with its greater knowledge and improved instruments has shown to be true —as if anyone with the time and the patience could not observe correctly! And he has been criticized for premature guesses on the basis of insufficient information, for his doctrines of purpose, for his denial of the atomic theory, and in general for having held back medieval scientists from more correct theories while they elaborated on his incorrect ones instead. But insufficient attention has been paid to the gigantic mental effort required to create order out of chaos, and to make the world *intelligible,* which was his primary purpose. No one before his time had seen the need for a method of inquiry, or classification of knowledge. Philosophers had speculated, and looked for universal principles, every now and then carrying out a few desultory experiments but always jumping to theoretical conclusions of little value beyond their aesthetic appeal. But to watch Aristotle at work trying to determine how to deal with zoology with no previous guide, as in the first book of his *Parts of Animals,* is to see the enormous difficulties that faced him in the struggle to put the material in order; and to read any part of the *Metaphysics* is to realize his extraordinary ability to handle the most difficult abstractions of thought with the utmost delicacy and sureness—in which again he had no predecessor. Plato charms us because of his artistry and imagination, and because there is no word that we cannot understand, no thought that we cannot follow. He flatters our ignorance, making us believe we are not as ignorant as we are; in reading Plato we all imagine ourselves philosophers. But Aristotle is hard work, and he makes no concessions to us; even when we think we have grasped one of his thoughts it quickly eludes us again. Then suddenly it becomes clear and fruitful and applicable in a hundred other ways, and we possess a tool for understanding the world.

In following the Aristotelian method as we have all followed it since his time without acknowledgments, our work has been made easy. But it was not easy for him. He had first to invent the tools of analysis, and then with these to set to work on all the phenomena of knowledge available to the Greek world. Both parts of his work he largely accomplished. His nephew went with Alexander on his expedition, and Alexander himself sent back data that he thought would be of interest to his old tutor. His students collected material for him, and he analyzed and classified it, no doubt with their assistance. For his *Politics* he analyzed and digested the constitutions of 158 different states, this analysis enabling him to classify the different kinds of states on the basis of evidence. He viewed the plays of his own age and the tragic drama of the great era, and in his *Poetics* classified the results, together with his findings in general terms of the requirements of tragedy. He did the same thing for the animal world in his three great works in zoology, the *History of Animals, Parts of Animals,* and *Generation of Animals;* and so on. Certainly in some cases he generalized and theorized too soon; but only very rarely did he fail to offer good reasons for the theories and for his acute criticisms of his predecessors. And never did his analysis fail. His successors could have built always upon his foundations, and revised his theories when necessary.

Summary of work of Aristotle—His place in the history of thought—It was a tragedy that Aristotle of all men should have been regarded as an authority and the last word on any subject, he who was the most ready of all the ancient investigators to base his theories on the observed facts. And it is now the prevalent opinion that when at last the late medieval scholars did begin to work on his findings at the University of Padua without accepting him as infallible, then they only had to revise his groundwork, and criticize some of his conclusions on the basis of their improved knowledge of the facts, and it was possible for Galileo, who studied at Padua, to lay the basis for modern science. Aristotle was not abandoned, save by the ignorant; but adapted, improved upon, and commented upon until at last he emerged as the great pioneer he was, but no longer "the master of those who know," which he was not.

If we examine the conclusions reached by Aristotle in all the numerous fields of inquiry to which he gave his attention, we shall find that they were almost always inspired by common sense, which has not been regarded as a useful tool in modern exact science with its powerful mathematics and instruments of research. Almost none of the findings of modern science, from the electron to the Copernican theory, from the physics of Einstein to the corpuscular-wave theory of light is validated by common sense or direct sense observation. For this reason Aristotle's conclusions in the physical sciences have to be interpreted very spaciously and charitably if they are to be in any way acceptable, while his conclusions in the social sciences may be as valid as in the days they were written.

Both Plato and Aristotle had an advantage over later thinkers in that the known world was small, and the whole range of knowledge was not very great. So it was still possible for one man to try to encompass it. Frequently throughout the work of Aristotle we find him making the statement that any science or art ought to cover the whole of a subject; and it is true that he makes the attempt. But not only this; he tries also to cover the whole of *all* subjects, using his key of logical analysis and systematic organization. This no successor has ever been able to do, and few have tried—though, as we shall see, it was the aim of Roger Bacon in medieval times. But even he did not find it necessary to go over a subject again once Aristotle had "completed" it; though toward the end of Bacon's life he suggested that a corps of specialists should be organized for the purpose of producing the necessary compendium. It is certain that no single person will ever try again.

This work of Aristotle was therefore unique, a last and most complete expression of the Greek desire for an orderly and harmonious whole, one of the greatest intellectual monuments in the history of mankind. If the highest praise is to be given, let us say that his work is worthy of the Greek genius.

Influence of Greek thought—Significance of Greek search for new truth

The great thinkers dealt with so far have occupied so much of the space in this chapter because they were the men whose thoughts provided the substratum for all the thinking of later Western man. The revolution ushered in by the Sophists has never been completed and perhaps never will be. At times, especially in the Middle Ages, men have preferred to take the traditional religiously inspired picture of the world as true and have not questioned its validity. This attitude has seldom led to new knowledge. The attitude of resignation in the face of divine will has sometimes prevailed in Western civilization, but always to the detriment of scientific inquiry. It might be more comfortable and give greater security to the individual to live in a world in which everything is known, and knowledge is contemplated, not enlarged; but such a world would be static. The world of the Sophists, in which one idea is as good as another, is a difficult world to live in, and it cannot be long endured. But the answer may well be that we need another Socrates to help us seek out the good, rather than despairing of finding it and resigning ourselves to the ethical nihilism which too often appears to present the only alternative to the acceptance of the teachings of tradition. The Greeks were the first to escape from the bonds imposed by their ancestors and strike out on a new path, the end of which could not and cannot now be seen. It is this above all that is meant by the Greek spirit. Greek art, perfect in its way as it is, has only been imitated by the West, copied but not equaled. For though we have inherited the Greek view of life and carried it on with our own genius to new realms unsuspected by the Greeks, the Greek feeling for man as a union of soul and body in equilibrium was peculiar to themselves, and we of the West can only dimly sense this view when we touch the few authentic masterpieces that have been preserved to us, and wonder at their perfection.

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Mathematical Platonism and its Opposites

Barry Mazur (Math professor at Harvard)

January 11, 2008

> We had the sky up there, all speckled with stars, and we used to lay on our backs and look up at them, and discuss about whether they was made or only just happened—Jim he allowed they was made, but I allowed they happened; I judged it would have took too long to make so many.

mused Huckleberry Finn. The analogous query that mathematicians continually find themselves confronted with when discussing their art with people who are not mathematicians is:

> Is mathematics discovered or invented?

I will refer to this as *The Question*, acknowledging that this five-word sentence, ending in a question-mark—and phrased in far less contemplative language than that used by Huck and Jim—may open conversations, but is hardly more than a token, standing for puzzlement regarding the status of mathematics.

One thing is—I believe—incontestable: if you engage in mathematics long enough, you bump into The Question, and it won't just go away[1]. If we wish to pay homage to the passionate felt experience that makes it so wonderful to think mathematics, we had better pay attention to it.

Some intellectual disciplines are marked, even scarred, by analogous concerns. Anthropology, for example has a vast, and dolefully introspective, literature dealing with the conundrum of whether we can ever avoid—wittingly or unwittingly—clamping the templates of our own culture onto whatever it is we think we are studying: how much are we discovering, how much inventing?

Such a *discovered/invented* perplexity may or may not be a burning issue for other intellectual pursuits, but it burns exceedingly bright for mathematics, and with a strangeness that isn't quite

[1]Garrison Keillor, a wonderful radio raconteur has in his repertoire a fictional character, Guy Noir, who tangles indefatigably with "life's persistent questions." This is all to the good. We should pay particular honor to the category of persistent questions even though—or, especially because—those are the chestnuts that we'll never crack.

matched when it pops up in other fields. For example, if you were to say—as Thomas Kuhn once did— "Priestley discovered oxygen but Lavoisier invented it" I think I know roughly what you mean by that utterance, without our having to synchronize our private vocabularies terribly much. But to intelligently comprehend each other's possibly differing attitudes towards circles, triangles, and numbers, we would also have to come to some—albeit ever-so-sketchy—understanding of how we each view, and talk about, a lot more than mathematics[2].

For me, at least, the anchor of any conversation about these matters is the experience of doing mathematics, and of groping for mathematical ideas. When I read literature that is ostensibly about The Question, I ask myself whether or not it connects in any way with my felt experience, and even better, whether it reveals something about it. I'm often—perhaps always—disappointed. The bizarre aspect of the mathematical experience—and this is what gives such fierce energy to The Question—is that one feels (I feel) that mathematical ideas can be hunted down, and in a way that is essentially different from, say, the way I am currently hunting the next word to write to finish this sentence. One can be a hunter and gatherer of mathematical concepts, but one has no ready words for the location of the hunting grounds. Of course we humans are beset with illusions, and the feeling just described could be yet another. There may be no location.

There are at least two standard ways of—if not exactly answering, at least—fielding The Question by offering a vocabulary of *location*. The colloquial tags for these locations are *In Here* and *Out There* (which seems to me to cover the field).

The first of these standard attitudes, the one with the logo *In Here*—which is sometimes called the Kantian (poor Kant!)—would place the source of mathematics squarely within our faculties of understanding. Of course faculties (Vermögen) and understanding (Verstand) are loaded eighteenth century words and it would be good —in this discussion at least—to disburden ourselves of their baggage as much as possible. But if this camp had to choose between discovery and invention, those two too-brittle words, it would opt for *invention.*

The "Out There" stance regarding the discovery/invention question whose heraldic symbol is Plato (poor Plato!) is to make the claim, starkly, that mathematics is the account we give of the timeless architecture of the cosmos. The essential mission, then, of mathematics is the accurate description, and exfoliation, of this architecture. This approach to the question would surely pick *discovery* over invention.

Strange things tend to happen when you think hard about either of these preferences.

For example, if we adopt what I labeled the Kantian position we should keep an eye on the stealth word "our" in the description of it that I gave, hidden as it is among behemoths of vocabulary (Vermögen, Verstand). Exactly *whose* faculties are being described? Who is the *we*? Is the *we* meant to be each and every one of us, given our separate and perhaps differing and often faulty faculties? If you feel this to be the case, then you are committed to viewing the mathematical enterprise to be as variable as humankind. Or are you envisioning some sort of distillate of all actual faculties, a more transcendental faculty, possessed by a kind of universal or ideal *we*, in

[2]For a start: you and I turn adjectives into nouns (*red* cows $\mapsto$ red; *five* cows $\mapsto$ five) with only the barest flick of a thought. What *is* that flick? Understanding the differences in our sense of what is happening here may tell us lots about our differences regarding matters that can only be discussed with much more mathematical vocabulary.

which case the Kantian view would seem to merge with the Platonic[3].

If we adopt the Platonic view that mathematics is discovered, we are suddenly in surprising territory, for this is a full-fledged theistic position. Not that it necessarily posits a god, but rather that its stance is such that the only way one can adequately express one's faith in it, the only way one can hope to persuade others of its truth, is by abandoning the arsenal of rationality, and relying on the resources of the prophets.

Of course, professional philosophers are in the business of formulating anti-metaphysical or metaphysical positions, decorticating them, defending them, and refuting them [4]. Mathematicians, though, may have another—or at least a prior—duty in dealing with The Question. That is, to be meticulous participant/observers, faithful to the one aspect of The Question to which they have sole proprietary rights: their own imaginative experience. What, precisely, describes our inner experience when we (and here the *we* is you and me) grope for mathematical ideas? We should ask this question open-eyed, allowing for the possibility that whatever it is we experience may delude us into fabricating ideas about some larger framework, ideas that have no basis[5].

I suspect that many mathematicians are as unsatisfied by much of the existent literature about The Question as I am. To be helpful here, I've compiled a list of Do's and Don't's for future writers promoting the Platonic or the Anti-Platonic persuasions.

- **For the Platonists.** One crucial consequence of the Platonic position is that it views mathematics as a project akin to physics, Platonic mathematicians being—as physicists certainly are—*describers* or possibly *predictors*—not, of course, of the physical world, but of some other more noetic entity. Mathematics—from the Platonic perspective—aims, among other things, to come up with the most faithful description of that entity.

  This attitude has the curious effect of reducing some of the urgency of that staple of mathematical life: *rigorous proof.* Some mathematicians think of mathematical proof as *the* certificate guaranteeing trustworthiness of, and formulating the nature of, the building-blocks of the edifices that comprise our constructions. Without proof: no building-blocks, no edifice. Our step-by-step articulated arguments are the devices that some mathematicians feel are responsible for bringing into being the theories we work in. This can't quite be so for the ardent Platonist, or at least it can't be so in the same way that it might be for the non-Platonist. Mathematicians often wonder about—sometimes lament—the laxity of proof in the physics literature. But I believe this kind of lamentation is based on a misconception, namely the misunderstanding of the fundamental function of *proof* in physics. Proof has principally (as

[3] A more general lurking question is exactly how we are to view the various ghosts in the machine of Kantian idealism—for example, who exactly *is* that little-described player haunting the elegant concept of *universally subjective judgments* and going under a variety of aliases: the *sensus communis* or the *allgemeine Stimme*?

[4] A very useful—and to my mind, fine—text that does exactly this type of lepidoptery is Mark Balaguer's *Platonism and Anti-Platonism in Mathematics*, Oxford Univ. Press (1998).

[5] When I'm working I sometimes have the sense—possibly the illusion—of gazing on the bare platonic beauty of structure or of mathematical objects, and at other times I'm a happy Kantian, marveling at the generative power of the intuitions for setting what an Aristotelian might call *the formal conditions of an object.* And sometimes I seem to straddle these camps (and this represents no contradiction to me). I feel that the intensity of this experience, the vertiginous imaginings, the leaps of intuition, the breathlessness that results from "seeing" but where the sights are of entities abiding in some realm of ideas, and the passion of it all, is what makes mathematics so supremely important for me. Of course, the realm might be illusion. But the experience?

it should have, in physics) a rhetorical role: to convince others that your description holds together, that your model is a faithful *re*-production, and possibly to persuade yourself of that as well. It seems to me that, in the hands of a mathematician who is a determined Platonist, proof could very well serve primarily this kind of rhetorical function—making sure that the description is on track—and not (or at least: not necessarily) have the rigorous theory-building function it is often conceived as fulfilling.

My feeling, when I read a Platonist's account of his or her view of mathematics, is that unless such issues regarding the nature of proof are addressed and conscientiously examined, I am getting a superficial account of the philosophical position, and I lose interest in what I am reading.

But the main task of the Platonist who wishes to persuade non-believers is to learn the trade, from prophets and lyrical poets, of how to communicate an experience that transcends the language available to describe it. If all you are going to do is to chant credos synonymous with "the mathematical forms are out there,"—which some proud essays about mathematical Platonism content themselves to do—well, that will not persuade.

- **For the Anti-Platonists.** Here there are many pitfalls. A common claim, which is meant to undermine Platonic leanings, is to introduce into the discussion the theme of *mathematics as a human, and culturally dependent pursuit* and to think that one is actually conversing about the topic at hand. Consider this, though: If the pursuit were *writing a description of the Grand Canyon* and if a Navajo, an Irishman, and a Zoroastrian were each to set about writing their descriptions, you can bet that these descriptions will be culturally-dependent, and even dependent upon the moods and education and the language of the three describers. But my having just recited all this relativism regarding the three descriptions does not undermine our firm faith in the *existence* of the Grand Canyon, their common focus. Similarly, one can be the most ethno-mathematically conscious mathematician on the globe, claiming that all our mathematical scribing is as contingent on ephemeral circumstance as this morning's rain, and *still* one can be the most devout of mathematical Platonists.

  Now this pitfall that I have just described is harmless. If I ever encounter this type of *mathematics is a human activity* argument when I read an essay purporting to defuse, or dispirit, mathematical Platonism I think to myself: human activity! what else could it be? I take this part of the essay as being irrelevant to The Question.

  A second theme that seems to have captured the imagination of some anti-Platonists is recent neurophysiological work— a study of blood flow into specific sections of the brain—as if this gives an *insider's view* of things[6]. Well, who knows? Neuro-anatomy and chemistry have been helpful in some discussions, and useless in others. To show this theme to be relevant would require a precisely argued explanation of exactly how *blood flow patterns* can refute, or substantiate, a Platonist—or any—-disposition. A satisfying argument of that sort would be quite a marvel! But just slapping the words *blood flow*—as if it were a poker-hand—onto a page doesn't really work.

  Sometimes the mathematical anti-Platonist believes that headway is made by showing Platonism to be unsupportable by rational means, and that it is an incoherent position to take when formulated in a propositional vocabulary.

  It is easy enough to throw together propositional sentences. But it is a good deal more difficult to capture a Platonic disposition in a propositional formulation that is a full and

[6]like the old Woody Allen movie *Everything you wanted to know about sex but were afraid to ask*

honest expression of some flesh-and-blood mathematician's view of things. There is, of course, no harm in trying—and maybe its a good exercise. But even if we cleverly came up with a proposition that *is* up to the task of expressing Platonism formally, the mere fact that the proposition cannot be demonstrated to be true won't necessarily make it vanish. There are many things—some true, some false— unsupportable by rational means. For example, if you challenge me to support—by rational means—my claim that I dreamt of Waikiki last night, I couldn't.

So, when is there harm? It is when the essayist becomes a *leveller.* Often this happens when the author writes extremely well, super coherently, slowly withering away the Platonist position by—well—the brilliant subterfuge of making the whole discussion boring, until I, the reader, become convinced—albeit momentarily, within the framework of my reading the essay—that there is no "big deal" here: the mathematical enterprise is precisely like any other cultural construct, and there is a fallacy lurking in any claim that it is otherwise. The Question is a non-question.

But someone who is not in love won't manage to definitively convince someone in love of the nonexistence of eros; so this mood never overtakes me for long. Happily I soon snap out of it, and remember again the remarkable sense of independence—autonomy even—of mathematical concepts, and the transcendental quality, the uniqueness—and the passion—of doing mathematics. I resolve then that (Plato or Anti-Plato) whatever I come to believe about The Question, my belief *must* thoroughly respect and not ignore all this.

Letters between Russell and Frege

Friday's Hill, Haslemere, 16 June 1902

Dear colleague,

For a year and a half I have been acquainted with your *Grundgesetze der Arithmetik*, but it is only now that I have been able to find the time for the thorough study I intended to make of your work. I find myself in complete agreement with you in all essentials, particularly when you reject any psychological element ⟦Moment⟧ in logic and when you place a high value upon an ideography ⟦Begriffsschrift⟧ for the foundations of mathematics and of formal logic, which, incidentally, can hardly be distinguished. With regard to many particular questions, I find in your work discussions, distinctions, and definitions that one seeks in vain in the works of other logicians. Especially so far as function is concerned (§ 9 of your *Begriffsschrift*), I have been led on my own to views that are the same even in the details. There is just one point where I have encountered a difficulty. You state (p. 17 ⟦p. 23 above⟧) that a function, too, can act as the indeterminate element. This I formerly believed, but now this view seems doubtful to me because of the following contradiction. Let w be the predicate: to be a predicate that cannot be predicated of itself. Can w be predicated of itself? From each answer its opposite follows. Therefore we must conclude that w is not a predicate. Likewise there is no class (as a totality) of those classes which, each taken as a totality, do not belong to themselves. From this I conclude that under certain circumstances a definable collection ⟦Menge⟧ does not form a totality.

I am on the point of finishing a book on the principles of mathematics and in it I should like to discuss your work very thoroughly.[1] I already have your books or shall buy them soon, but I would be very grateful to you if you could send me reprints of your articles in various periodicals. In case this should be impossible, however, I will obtain them from a library.

The exact treatment of logic in fundamental questions, where symbols fail, has remained very much behind; in your works I find the best I know of our time, and therefore I have permitted myself to express my deep respect to you. It is very regrettable that you have not come to publish the second volume of your *Grundgesetze*; I hope that this will still be done.

Very respectfully yours,

BERTRAND RUSSELL

The above contradiction, when expressed in Peano's ideography, reads as follows:

$$w = \mathrm{cls} \cap x \ni (x \sim\varepsilon\, x) . \supset : w\, \varepsilon\, w . = . w \sim\varepsilon\, w.$$

I have written to Peano about this, but he still owes me an answer.

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Jena, 22 June 1902

Dear colleague,

Many thanks for your interesting letter of 16 June. I am pleased that you agree with me on many points and that you intend to discuss my work thoroughly. In response to your request I am sending you the following publications:

1. "Kritische Beleuchtung" ⟦*1895*⟧,
2. "Ueber die Begriffsschrift des Herrn Peano" ⟦*1896*⟧,
3. "Ueber Begriff und Gegenstand" ⟦*1892*⟧,
4. "Über Sinn und Bedeutung" ⟦*1892a*⟧,
5. "Ueber formale Theorien der Arithmetik" ⟦*1885*⟧.

I received an empty envelope that seems to be addressed by your hand. I surmise that you meant to send me something that has been lost by accident. If this is the case, I thank you for your kind intention. I am enclosing the front of the envelope.

When I now read my *Begriffsschrift* again, I find that I have changed my views on many points, as you will see if you compare it with my *Grundgesetze der Arithmetik*. I ask you to delete the paragraph beginning "Nicht minder erkennt man" on page 7 of my *Begriffsschrift* ⟦"It is no less easy to see", p. 15 above⟧, since it is incorrect; incidentally, this had no detrimental effects on the rest of the booklet's contents.

Your discovery of the contradiction caused me the greatest surprise and, I would almost say, consternation, since it has shaken the basis on which I intended to build arithmetic. It seems, then, that transforming the generalization of an equality into an equality of courses-of-values ⟦die Umwandlung der Allgemeinheit einer Gleichheit in eine Werthverlaufsgleichheit⟧ (§ 9 of my *Grundgesetze*) is not always permitted, that my Rule V (§ 20, p. 36) is false, and that my explanations in § 31 are not sufficient to ensure that my combinations of signs have a meaning in all cases. I must reflect further on the matter. It is all the more serious since, with the loss of my Rule V, not only the foundations of my arithmetic, but also the sole possible foundations of arithmetic, seem to vanish. Yet, I should think, it must be possible to set up conditions for the transformation of the generalization of an equality into an equality of courses-of-values such that the essentials of my proofs remain intact. In any case your discovery is very remarkable and will perhaps result in a great advance in logic, unwelcome as it may seem at first glance.

Incidentally, it seems to me that the expression "a predicate is predicated of itself" is not exact. A predicate is as a rule a first-level function, and this function requires an object as argument and cannot have itself as argument (subject). Therefore I would prefer to say "a concept is predicated of its own extension". If the function $\Phi(\xi)$ is a concept, I denote its extension (or the corresponding class) by "$\grave{\varepsilon}\Phi(\varepsilon)$" (to be sure, the justification for this has now become questionable to me). In "$\Phi(\grave{\varepsilon}\Phi(\varepsilon))$" or "$\grave{\varepsilon}\Phi(\varepsilon) \cap \grave{\varepsilon}\Phi(\varepsilon)$"[1] we then have a case in which the concept $\Phi(\xi)$ is predicated of its own extension.

The second volume of my *Grundgesetze* is to appear shortly. I shall no doubt have to add an appendix in which your discovery is taken into account. If only I already had the right point of view for that!

Very respectfully yours,

G. FREGE

From Reuben Hersh's book, "What is Mathematics, Really?"

Immanuel Kant (1724–1804)

Synthetic a priori. Non-Euclidean Geometry.

Classical philosophy reached its peak at the end of the eighteenth century in Kant. Kant's metaphysics is a continuation of the Platonic search for certainty and timelessness in human knowledge. He wanted to rebut Hume's denial of certainty. To do so, he made a sharp distinction between noumena, things in themselves, which we can never know, and phenomena, appearances, which our senses tell us. His goal was knowledge a priori—knowledge timeless and independent of experience.

He distinguished two kinds of a priori knowledge. The "analytic a priori" is the kind we know by logical analysis, by the meanings of the terms being used. Like the rationalists, Kant believed we also possess a priori knowledge that is not logical truism. This is his "synthetic a priori." Our intuitions of time and space are such knowledge, he believed. He explained their a priori nature by saying they're intuitions—inherent properties of the human mind. Our intuition of time is systematized in arithmetic, based on the intuition of *succession*. Our intuition of space is systematized in geometry. For Kant, as for all earlier thinkers, there's only one geometry—the one we call Euclidean. The truths of geometry and arithmetic are forced on us by the way our minds work; this explains why they are (supposedly) true for everyone, independent of experience. The intuitions of time and space, on which arithmetic and geometry are based, are objective in the sense that they're valid for every human mind. No claim is made for existence outside the human mind. Yet the Euclid myth (see below) remains central in Kantian philosophy.

Indeed, mathematics is central for Kant. His *Prolegomena to any Future Metaphysics Which Will Be Able to Come Forth as a Science*, has three parts. Part One is, "How Is Pure Mathematics Possible?" (a question I discussed in Chapter 1).

Kant's fundamental presupposition is that "contentful knowledge independent of experience (the 'synthetic a priori') can be established on the basis of universal human intuition." In *The Critique of Pure Reason*, he gives the two examples already mentioned: (1) space intuition, the foundation of geometry, and (2) time intuition, the foundation of arithmetic. In *The Critique of Practical Reason*, without using the term "synthetic a priori," he gives a third intuition: (3) moral intuition, the foundation of religion.

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I will discuss intuitions (1) and (3), although (2) is still interesting today. It deals with the question, "Are primitive counting notions universal and invariable?" Most writers think so. Wittgenstein disagreed. Frege also disagreed; he thought arithmetic was analytic a priori—based on logic—rather than synthetic a priori—based on time intuition.

I discuss three topics on Kant:

1. Synthetic a priori. Intuition of space and time.
2. Effect of non-Euclidean geometry on Kant's theory of space intuition.
3. Intuition of duty, God, and the parallel with intuitions of space and time.

1. The *Prolegomena*, p. 21: "Weary therefore of dogmatism [Leibniz], which teaches us nothing, and of skepticism [Hume], which does not even promise us anything—even the quiet state of a contented ignorance—disquieted by the importance of knowledge so much needed, and rendered suspicious by long experience of all knowledge which we believe we possess or which offers itself in the name of pure reason, there remains but one critical question, on the answer to which our future procedure depends, namely, "Is metaphysics at all possible?". . . The *Prolegomena* must therefore rest upon something already known as trustworthy, from which we can set out with confidence and ascend to sources as yet unknown, the discovery of which will not only explain to us what we knew but exhibit a sphere of many cognitions which all spring from the same sources. The method of prolegomena, especially of those designed as a preparation for future metaphysics, is consequently analytical.

"But it happens, fortunately, that though we cannot assume metaphysics to be an actual science, we can say with confidence that there is actually given certain pure *a priori* synthetical cognitions, pure mathematics and pure physics; for both contain propositions which are unanimously recognized, partly apodictically certain by mere reason, partly by general consent arising from experience and yet as independent of experience. We have therefore at least some uncontested synthetical knowledge *a priori*, and need not ask *whether* it be possible, for, it is actual, but *how* it is possible, in order that we deduce from the principle which makes the given knowledge possible the possibility of the rest."

Against synthetic (contentful) knowledge he contrasted analytic knowledge, which is derived from logic and the meaning of words.

Again from the *Prolegomena*:

"Analytical judgments express nothing in the predicate but what has been already actually thought in the concept of the subject, though not so distinctly or with the same (full) consciousness. When I say, 'All bodies are extended,' I have not amplified in the least my concept of body, but have only analyzed it, as extension was really thought to belong to that concept before the judgment was made, though it was not expressed. This judgment is therefore analytical. On the contrary, this judgment, "All bodies have weight," contains in its predicate something not actually thought in the universal concept of body. It amplifies my knowledge by adding something to my concept, and must therefore be called synthetical.

"First of all, we must observe that all strictly mathematical judgments are a priori, and not empirical, because they carry with them necessity, which cannot be obtained from experience. . . . It must at first be thought that the proposition $7 + 5 = 12$ is a mere analytical judgment, following from the concept of the sum, of seven and five, according to the law of contradiction. But on close examination it appears that the concept of the sum of $7 + 5$ contains merely their union in a single number . . . Just as little is any principle of geometry analytical." (This is the point at which Frege turned away from Kant.)

Richard Tarnas writes (p. 342): "The clarity and strict necessity of mathematical truth had long provided the rationalists—above all Descartes, Spinoza and Leibniz—with the assurance that, in the world of modern doubt the human mind had at least one solid basis for attaining certain knowledge. Kant himself had long been convinced that natural science was scientific to the precise extent that it approximated to the ideal of mathematics. . . . By Hume's reasoning, with which Kant had to agree, the certain laws of Euclidean geometry could not have been derived from empirical observation. Yet Newtonian science was explicitly based upon Euclidean geometry. . . . Kant began by noting that if all content that could be derived from experience was withdrawn from mathematical judgments, the ideas of space and time will remain. From this he inferred that any event experienced by the senses is located automatically in a framework of spatial and temporal relations. Space and time are 'a priori forms of human sensibility': They condition whatever is apprehended through the senses. Mathematics could accurately describe the empirical world because mathematical principles necessarily involve a context of space and time, and space and time lay at the basis of all sensory experience: they condition and structure any empirical observation. . . . Because [geometrical] propositions are based on direct intuitions of spatial relations, they are '*a priori*'—constructed by the mind and not derived from experience—and yet they are also valid for experience, which will by necessity conform to the *a priori* form of space."

Kant's intuitions are supposed to explain, not how we might or could, but how we *actually do* conceive of time and space. There's no claim that they correspond to an objective reality. They're properties of Mind.

For Kant and his predecessors, mathematics and Mind are unchanging, eternal, and universal. Kant's intuitions are supposed to be eternal, universal features of Mind. But the Mind Kant knows is the mind of eighteenth-century Europe, plus the books in his library. He assumes this constitutes all human thinking.

2. Kant's views came to dominate West European philosophy, in spite of a development in geometry that made Kant's account of space untenable. That development was non-Euclidean geometry.**

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The fifth axiom of Euclid's *Elements*, the parallel postulate, for centuries was considered a blot on the fair cheek of geometry. This postulate says: "If a line A crossing two lines B and C makes the sum of the interior angles on one side of A less than two right angles, then B and C meet on that side."

An equivalent axiom, the usual one in geometry books, is Playfair's: "Through a point not on a given line passes one parallel to the line."

This parallel axiom, everybody agreed, is intuitively true. Yet it isn't as "self-evident" as the other axioms. It says something happens at a point that possibly is very remote, where our intuition isn't as firm as nearby. Mathematicians wanted it proved, not assumed as Euclid did. Many tried, no one succeeded.

Then, as I mentioned in Chapter 4, Gauss, Bolyai, and Lobachevsky had the same brilliant idea: Suppose the fifth postulate *is* false, and then see what happens! Each of them got a new geometry! A possibility never before conceived.

Later Beltrami, Klein, and Poincaré showed that Euclidean and non-Euclidean geometry are "equiconsistent." If either is consistent, so is the other. Since no one doubts that Euclidean geometry is consistent, non-Euclidean also is believed to be consistent.

Kant's theory of spatial intuition meant Euclidean geometry was inescapable. But the establishment of non-Euclidean geometry gives us choices. Which geometry works best in physics? The question becomes empirical, to be settled by observation.

In 1915, Einstein published his theory of general relativity. The cosmos is a non-Euclidean curved space-time, more general than the hyperbolic space of Gauss, Lobatchevsky, and Bolyai. So non-Euclidean geometry is not just consistent, it governs the universe! Non-Euclidean geometry is used to represent relativistic velocity vectors. Physics doesn't prefer Euclid to non-Euclid.

Our intuitive notion of space is learned on a small scale, compared to the universe as a whole. Locally, "in the small," the difference between Euclidean and non-Euclidean geometries is too tiny to notice. The belief that the Euclidean angle sum theorem is "indubitable" or "absolute" is based on belief in an infallible spatial intuition. That belief is discredited by non-Euclidean geometry and general relativity.

Decades before non-Euclidean geometry was discovered by Kant's countryman Karl Friedrich Gauss, it was "almost known" to Johann Heinrich Lambert (1728–1777), a German mathematician who was actually an acquaintance or friend of Kant! Lambert came to a crucial recognition—that *if* the "postulate of the acute angle" were true it would lead to a strange new geometry. This already would have refuted Kant's theory that Euclidean geometry is an unavoidable innate intuition of the human mind.

Did Kant know Lambert's work? Martin thinks he did, but disregarded it as a "mere abstraction."

Körner says Kant didn't deny the abstract conceivability of non-Euclidean geometries; he thought they could never be realized in real time and space. This idea was wiped out by the advance of science.

Even though Kant's philosophy of space had already been exploded by non-Euclidean geometry, Philip Kitcher shows that all three foundationist gurus—Frege, Hilbert, and Brouwer—were Kantians. That was a consequence of the dominance of Kantianism in their early milieus, and the usual tendency of research mathematicians toward an idealist viewpoint. When they became disturbed by the "crisis in foundations" they couldn't help thinking in Kant's categories, in particular, his analytic and synthetic a priori. But instead of talking about the synthetic a priori, they talked about restoring the indubitability of mathematics—building or finding a solid foundation.

Non-Euclidean geometry makes Kant's philosophy of space untenable. But mathematicians avoid philosophical disputation by not mentioning the issue. To this day, texts on non-Euclidean geometry ignore its revolutionary philosophical implications. The first direct statement of the contradiction seems to be by Hermann Helmholtz, in *Mind* in 1877 (the birth year of that august journal.) In the next volume of *Mind* a Dutch philosopher, H. K. Land, replied that, by the nature of things, nothing in mathematics could be relevant to Kant's theory. Modern philosophy texts and lecturers on Kant seem to follow Land's principle. They don't mention non-Euclidean geometry.

3. Kant may have been the last philosopher or mathematician in the chain from Pythagoras to the present who explicitly made theology part of his philosophy. There's a half-hidden connection between Kant's a prioristic philosophy of mathematics and his moral-intuition version of Christianity.

In the *Critique of Practical Reason* he demolishes the three standard proofs of the existence of God. "Ontological": By definition, God is Perfect. Nonexistence would be an imperfection. "Cosmological": Every event has a cause. To avoid infinite regress, there had to have been a First Cause (God). "Teleological": A watch has a watch-maker. The World is more intricate than a watch, so it has a World-Maker (God).

Kant tears these proofs to shreds. He says they're the only proofs "speculative reason" (Leibnizian rationalism) could ever give. Kant isn't doubting God's existence. He's showing the superiority of his own proof, based on intuition. Not so different from his intuitions of time and space. Everyone has an intuition of duty, Kant thinks, of right and wrong. He doesn't say this *proves* God exists. He says it *justifies the postulate* "God exists."

"The moral law leads us to postulate not only the immortality of the soul, but the existence of God. . . . This second postulate of the existence of God rests upon the necessity of presupposing the existence of a cause adequate to the effect which has to be explained . . . a being who is a part of the world and is dependent upon it . . . ought to seek to promote the highest good, and therefore the highest

good must be possible. . . . There is therefore implied, in the idea of the highest good, a being who is the supreme cause of nature, and who is the cause or author of nature through his intelligence and will, that is, God . . . or, in other words, it is morally necessary to hold the existence of God."

And in the *Prolegomena*, paras. 354–55, p. 103: "We must therefore think an immaterial being, a world of understanding, and a Supreme Being (all mere noumena) because in them only, as things in themselves, reason finds that completion and satisfaction which it can never hope for in the derivation of appearances from their homogeneous grounds, and because these actually have reference to something distinct from them (and totally heterogeneous), as appearances always presuppose an object in itself, and therefore suggest its existence whether we can know more of it or not."

Tarnas again (p. 350): "It is clear that at heart Kant believed that the laws moving the planets and stars ultimately stood in some fundamental harmonious relation to the moral imperatives he experienced within himself. 'Two things fill the heart with ever new and always increasing awe and admiration: the starry heavens above me and the moral law within me.' But Kant also knew he could not prove that relation, and in his delimitation of human knowledge to appearances, the Cartesian schism between the human mind and the material cosmos continued in a new and deepened form.

"In the subsequent course of Western thought, it was to be Kant's fate that, as regards both religion and science, the power of his epistemological critique tended to outweigh his positive affirmations. On the one hand, the room he made for religious belief began to resemble a vacuum, since religious faith had now lost any external support from either the empirical world or pure reason, and increasingly seemed to lack internal plausibility and appropriateness for secular modern man's psychological character. On the other hand, the certainty of scientific knowledge, already unsupported by any external mind-independent necessity after Hume and Kant, became unsupported as well by any internal cognitive necessity with the dramatic controversion by twentieth century physics of the Newtonian and Euclidean categories which Kant had assumed were absolute" (Tarnas, p. 350).

As the universal intuition of space is refuted by non-Euclidean geometry, the universal intuition of duty is refuted by history. For Winston Churchill and Harry Truman, fire-bombing German and Japanese civilians was duty. In the police stations of the world, torturing prisoners is duty. In Nazi Germany, genocide was duty.

What's the connection between Kant's philosophy of mathematics and his moral-intuition version of religion? Unlike Descartes and Leibniz, Kant does not use the certainty of mathematics (time and space) to support the certainty of God's existence. He considers the intuition of duty independently of the intuitions of time or space. He keeps his theory of God separate from his theory of mathematics. But they both have the same logic. Both rely on intuition: knowledge coming, not from the senses, study, or learning, but from the nature of Mind. Right and wrong, like time and space, are universal intuitions. Our space intuition leads to geometry, our time intuition leads to arithmetic, our duty intuition leads to Divinity.

In God's mind, the difficulties and puzzles in philosophy of mathematics disappear. How do numbers exist? Why do mathematical facts seem certain and timeless? Why does mathematics work in the "real world"?

In the mind of God, it's no problem.

The trouble with today's Platonism is that it gives up God, but wants to keep mathematics a thought in the mind of God.

Euclid as a Myth. Nobody's Perfect.

The myth of Euclid is the belief that Euclid's *Elements* contain indubitable truths about the universe. Even today, most educated people still believe the Euclid myth. Up to the middle or late nineteenth century, the myth was unquestioned. It has been the major support for metaphysical philosophy—philosophy that sought a priori certainty about the nature of reality.

The roots of our philosophy of mathematics are in classical Greece. For the Greeks, mathematics was geometry. In Plato and Aristotle, philosophy of mathematics is philosophy of geometry.

Rationalism served science by denying the intellectual supremacy of religious authority, while defending the truth of religion. This equivocation gave science room to grow without being strangled as a rebel. It claimed for science the right to independence from the Church. Yet this independence didn't threaten the Church, since science was the study of God's handiwork. "The heavens proclaim the glory of God and the firmament showeth His handiwork."

The existence of mathematical objects as ideas independent of human minds was no problem for Newton or Leibniz; they took for granted the existence of a Divine Mind. In that belief, the problem is rather to account for the existence of nonideal, material objects.

After rationalism displaced medieval scholasticism, it was challenged by materialism and empiricism; by Locke and Hobbes in Britain, by the encyclopedists in France. The advance of science on the basis of the experimental method gave the victory to empiricism. The conventional wisdom became: "The material universe is the fundamental reality. Experiment and observation are the only legitimate means of studying it."

The empiricists held that all knowledge *except mathematical* comes from observation. They usually didn't try to explain how mathematical knowledge originates. In the controversies, first between rationalism and scholasticism, later between rationalism and empiricism, the sanctity of geometry was unchallenged.

Philosophers disputed whether we proceed from Reason (a gift from the Divine) to discover the properties of the world, or whether only our bodily senses can do so. Both sides took it for granted that geometrical knowledge is not problematical, even if all other knowledge is. Hume exempted books of mathematics and of natural science from his outcry, "Commit it to the flames."

For rationalists, mathematics was the main example to confirm their view of the world. For empiricists, it was an embarrassing counter-example, which had to be ignored or explained away. If, as seemed obvious, mathematics contains knowledge independent of sense perception, then empiricism is inadequate as an explanation of all human knowledge. This embarrassment is still with us; it's a reason for the difficulties of philosophy of mathematics.

Mathematics always had a special place in the battle between rationalism and empiricism. The mathematician-in-the-street, with his common-sense belief in mathematics as knowledge, is the last vestige of rationalism.

The modern scientific outlook took ascendancy in the nineteenth century. By the time of Russell and Whitehead, only logic and mathematics could still claim to be nonempirical knowledge, obtained directly by Reason.

From the customary viewpoint among scientists now, the Platonism of most mathematicians is an anomaly. For many years the accepted assumptions in science have been materialism in ontology, empiricism in epistemology. The world is all one stuff, "matter," which physics studies. If matter gets into certain complicated configurations, it falls under a special science with its own methodology—chemistry, geology, and biology. We learn about the world by looking at it and thinking about what we see. Until we look, we have nothing to think about.

Yet in mathematics we have knowledge of things we can never observe. At least, this is the natural point of view when we aren't trying to be philosophical.

Until well into the nineteenth century, the Euclid myth was universal among mathematicians as well as philosophers. Geometry was the firmest, most reliable branch of knowledge. Mathematical analysis—calculus and its extensions and ramifications—derived legitimacy from its link with geometry. We needn't say "Euclidean geometry." The qualifier became necessary only after non-Euclidean geometry had been recognized. Before that, geometry was simply geometry—the study of the properties of space. These were exact, eternal, and knowable with certainty by the human mind.

Intuitionistic reflections on formalism

LUITZEN EGBERTUS JAN BROUWER

(*1927a*)

While logicism and intuitionism were too far apart to allow a dialogue between them, the emergence of Hilbert's metamathematics created between Hilbert and Brouwer a ground on which a discussion could proceed, however deep might be the disagreement between these two mathematicians on the role of consistency proofs. In 1912 Brouwer ended a presentation of intuitionism (*1912*, or *1912a*) on a pessimistic note, despairing of any communication between two groups of scholars who were not speaking the same tongue and could not learn each other's tongue. In the text below, which is § 1 of *1927a*, Brouwer lists four points concerning which he considers that intuitionism and formalism could enter into a dialogue. This, it seems, could be true of the first three points, which bring out the similarities between finitary metamathematics and a certain part of intuitionistic mathematics, but could hardly be true of the fourth point, which states that consistency proofs are unable to provide a foundation for mathematics. There is a commentary on § 1 of Brouwer's paper in *Heyting 1934*, pp. 54–57, or *Heyting 1955*, pp. 60–63.

The omitted § 2, which has no direct connection with § 1, investigates various intuitionistic versions of the principle of excluded middle as well as the conditions in which each of these versions is applicable; Brouwer returned to this question in *1948*, pp. 1243–1248, and *1953*, pp. 3–5.

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The disagreement over which is correct, the formalistic way of founding mathematics anew or the intuitionistic way of reconstructing it, will vanish, and the choice between the two activities be reduced to a matter of taste, as soon as the following insights, which pertain primarily to formalism but were first formulated in the intuitionistic literature, are generally accepted. The acceptance of these insights is only a question of time, since they are the results of pure reflection and hence contain no disputable element, so that anyone who has once understood them must accept them. Two of the four insights have so far been understood and accepted in the formalistic literature. When the same state of affairs has been reached with respect to the other two, it will mean the end of the controversy concerning the foundations of mathematics.

First insight. *The differentiation, among the formalistic endeavors, between a construction of the "inventory of mathematical formulas" (formalistic view of mathematics) and an intuitive (contentual) theory of the laws of this construction, as well as the*

recognition of the fact that for the latter theory the intuitionistic mathematics of the set of natural numbers is indispensable.

Second insight. *The rejection of the thoughtless use of the logical principle of excluded middle, as well as the recognition, first, of the fact that the investigation of the question why the principle mentioned is justified and to what extent it is valid constitutes an essential object of research in the foundations of mathematics, and, second, of the fact that in intuitive (contentual) mathematics this principle is valid only for finite systems.*

Third insight. *The identification of the principle of excluded middle with the principle of the solvability of every mathematical problem.*

Fourth insight. *The recognition of the fact that the (contentual) justification of formalistic mathematics by means of the proof of its consistency contains a vicious circle, since this justification rests upon the (contentual) correctness of the proposition that from the consistency of a proposition the correctness of the proposition follows, that is, upon the (contentual) correctness of the principle of excluded middle.*

1. The first insight is still lacking in *Hilbert 1904*, see in particular Section V, pp. 184–185 [[above, pp. 137–138]], which is in contradiction with it. After having been strongly prepared by Poincaré, it first appears in the literature in *Brouwer 1907*, where on pp. 173–174 the terms *mathematical language* and *mathematics of the second order* are used to distinguish between the parts of formalistic mathematics mentioned above and where the intuitive character of the latter part is emphasized.[1] This insight penetrated into the formalistic literature with *Hilbert 1922* (see in particular p. 165 and p. 174), where mathematics of the second order was given the name *metamathematics*. The claim of the formalistic school to have reduced intuitionism to absurdity by means of this insight, borrowed from intuitionism, is presumably not to be taken seriously.

2. The thoughtless use of the logical principle of excluded middle is still to be found in *Hilbert 1904* and *1917* (see, for example, *1917*, p. 413, ll. 11u–4u, and in particular *1904*: p. 182, ll. 16–19; p. 182, l. 2u, to p. 183, l. 2; p. 184, ll. 21u–13u [[above, p. 135, ll. 13u–11u; p. 136, ll. 5–7; p. 137, ll. 13–18]]; in each of these places the principle of excluded middle is regarded as essentially equivalent to the principle of contradiction). The second insight is found in the literature for the first time in *Brouwer 1908* and then at greater or lesser length in *Brouwer 1912, 1914, 1917, 1919b, 1923b*, and *1923d*. Except for the recognition, most intimately connected with it, of the intuitionistic consistency of the principle of excluded middle, it penetrates the formalistic literature with *Hilbert 1922a*,[2] where, on the one hand, the limited contentual validity of the principle of excluded middle is acknowledged (see in particular pp. 155–156) and, on the other, the task is posed of consistently combining a logical formulation of the principle of excluded middle with other axioms in the framework of formalistic mathematics. The limited contentual validity of the principle of excluded middle is pointed out with particular eloquence in *Hilbert 1925* (pp. 173–174 [[above, pp. 378–379]]), where, however, the goal is overshot when the area called into question is extended to include the remaining Aristotelian laws.

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[1] An oral discussion of the first insight took place in several conversations I had with Hilbert in the autumn of 1909.

[2] After attention had already been paid to the principle of excluded middle in *Hilbert 1922*, p. 160.

3. During the period of the thoughtless use of the principle of excluded middle in the formalistic literature, the principle of the solvability of every mathematical problem is first advanced in *Hilbert 1900b*, p. 52, as an axiom or a conviction and then in *Hilbert 1917*, pp. 412–413, in two different forms (in which, instead of "solvability", "solvability in principle" and, after that, "decidability by means of a finite number of operations" are mentioned) as the object of problems still to be settled. But even after the discussion of the third insight in *Brouwer 1908*, p. 156, *1914*, p. 80, *1919b*, pp. 203–204, and the penetration of the second insight into the formalistic literature, we find that in *Hilbert 1925*, p. 180 [[above, p. 384]]—where the problem of the consistency of the axiom of the solvability of any mathematical problem is offered as an example of a "problem of a fundamental character that falls within the domain of mathematics but formerly could not even be approached"—this question is presented as still open, irrespective of whether the foundations of the science of mathematics (which also comprise the consistency of the principle of excluded middle) be secured or not.

4. The fourth insight is expressed in *Brouwer 1927*, p. 64 [[above, p. 460]]. No trace of it is to be found thus far in the formalistic literature but many an utterance contradicting it, for example in *Hilbert 1900b*, pp. 55–56, and above all in *Hilbert 1925*, where on pp. 162–163 [[above, p. 370]] we still find the exclamation: "No, if justifying a procedure means anything more than proving its consistency, it can only mean determining whether the procedure is successful in fulfilling its purpose."

According to what precedes, formalism has received nothing but benefactions from intuitionism and may expect further benefactions. The formalistic school should therefore accord some recognition to intuitionism, instead of polemicizing against it in sneering tones while not even observing proper mention of authorship. Moreover, the formalistic school should ponder the fact that in the framework of formalism *nothing* of mathematics proper has been secured up to now (since, after all, the metamathematical proof of the consistency of the axiom system is lacking, now as before), whereas intuitionism, on the basis of its constructive definition of set[3] and the fundamental property it has exhibited for finitary sets,[4] has already erected anew several of the theories of mathematics proper in unshakable certainty. If, therefore, the formalistic school, according to its utterance in *Hilbert 1925*, p. 180 [[above, p. 384]], has detected modesty on the part of intuitionism, it should seize the occasion not to lag behind intuitionism with respect to this virtue.

[3] [[Later Brouwer uses the word "spread" for this notion; here the word "Menge", translated as "set", suggests that Brouwer considers spreads to be constructive substitutes for classical sets. See above, p. 453.]]

[4] See *Brouwer 1927*, p. 66, Theorem 2 [[above, p. 462]].

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On the infinite

DAVID HILBERT

(1925)

Weierstrass, through a critique elaborated with the sagacity of a master, created a firm foundation for mathematical analysis. By clarifying, among other notions, those of minimum, function, and derivative, he removed the remaining flaws from the calculus, cleansed it of all vague ideas concerning the infinitesimal, and conclusively overcame the difficulties that until then had their roots in the notion of infinitesimal. If today there is complete agreement and certitude in analysis whenever modes of inference are employed that rest upon the notions of irrational number and of limit in general, and if there is unanimity on all results concerning the most complicated questions in the theory of differential and integral equations, despite the boldest use of the most diverse combinations of superposition, juxtaposition, and nesting of limits, this is essentially due to the scientific activity of Weierstrass.

Nevertheless, the discussions about the foundations of analysis did not come to an end when Weierstrass provided a foundation for the infinitesimal calculus.

The reason for this is that the significance of the *infinite* for mathematics had not yet been completely clarified. To be sure, the infinitely small and the infinitely large were eliminated from analysis, as established by Weierstrass, through a reduction of the propositions about them to [[propositions about]] relations between finite magnitudes. But the infinite still appears in the infinite number sequences that define the real numbers, and, further, in the notion of the real number system, which we conceive to be an actually given totality, complete and closed.

The forms of logical inference in which this conception finds its expression—namely, those that we employ when, for example, we deal with *all* real numbers having a certain property or assert that *there exist* real numbers having a certain

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property—are called upon quite without restriction and are used again and again by Weierstrass precisely when he is establishing the foundations of analysis.

Thus the infinite, in a disguised form, was able to worm its way back into Weierstrass' theory and escape the sharp edge of his critique; therefore it is the *problem of the infinite* in the sense just indicated that still needs to be conclusively resolved. And just as the infinite, in the sense of the infinitely small and the infinitely large, could, in the case of the limiting processes of the infinitesimal calculus, be shown to be a mere way of speaking, so we must recognize that the infinite in the sense of the infinite totality (wherever we still come upon it in the modes of inference) is something merely apparent. And just as operations with the infinitely small were replaced by processes in the finite that have quite the same results and lead to quite the same elegant formal relations, so the modes of inference employing the infinite must be replaced generally by finite processes that have precisely the same results, that is, that permit us to carry out proofs along the same lines and to use the same methods of obtaining formulas and theorems.

That, then, is the purpose of my theory. Its aim is to endow mathematical method with the definitive reliability that the critical era of the infinitesimal calculus did not achieve; thus it shall bring to completion what Weierstrass, in providing a foundation for analysis, endeavored to do and toward which he took the first necessary and essential step.

But in clarifying the notion of the infinite we must still take into consideration a more general aspect of the question. If we pay close attention, we find that the literature of mathematics is replete with absurdities and inanities, which can usually be blamed on the infinite. So, for example, some stress the stipulation, as a kind of restrictive condition, that, if mathematics is to be rigorous, only a *finite* number of inferences is admissible in a proof—as if anyone had ever succeeded in carrying out an infinite number of them!

Even old objections that have long been regarded as settled reappear in a new guise. So in recent times we come upon statements like this: even if we could introduce a notion safely (that is, without generating contradictions) and if this were demonstrated, we would still not have established that we are justified in introducing the notion. Is this not precisely the same objection as the one formerly made against complex numbers, when it was said that one could not, to be sure, obtain a contradiction by means of them, but their introduction was nevertheless not justified, for, after all, imaginary magnitudes do not exist? No, if justifying a procedure means anything more than proving its consistency, it can only mean determining whether the procedure is successful in fulfilling its purpose. Indeed, success is necessary; here, too, it is the highest tribunal, to which everyone submits.

Another author seems to see contradictions, like ghosts, even when nothing has been asserted by anyone at all, namely, in the concrete world of perception [[Sinnenwelt]] itself, whose "consistent functioning" is regarded as a special assumption. I, for one, have always believed that only assertions and, insofar as they lead to assertions by means of inferences, assumptions could contradict each other, and the view that facts and events themselves could come to do so seems to me the perfect example of an inanity.

By these remarks I wanted to show only that the definitive clarification of the *nature of the infinite* has become necessary, not merely for the special interests of the individual sciences, but rather for the *honor of the human understanding* itself.

The infinite has always stirred the *emotions* of mankind more deeply than any other question; the infinite has stimulated and fertilized reason as few other *ideas* have; but also the infinite, more than any other *notion*, is in need of *clarification*.

If we now turn to this task, to the clarification of the nature of the infinite, we must ever so briefly call to mind the contentual significance that attaches to the infinite in reality; first we see what we can learn about this from physics.

The initial, naive impression that we have of natural events and of matter is one of uniformity, of continuity. If we have a piece of metal or a volume of liquid, the idea impresses itself upon us that it is divisible without limit, that any part of it, however small, would again have the same properties. But, wherever the methods of research in the physics of matter were refined sufficiently, limits to divisibility were reached that are not due to the inadequacy of our experiments but to the nature of the subject matter, so that we could in fact view the trend of modern science as an emancipation from the infinitely small and, instead of the old maxim "natura non facit saltus", now assert the opposite, "nature makes leaps".

As is well known, all matter is composed of small building blocks, *atoms*, which, when combined and connected, yield the entire multiplicity of macroscopic substances.

But physics did not stop at the atomic theory of matter. Toward the end of the last century the atomic theory of electricity, which at first seemed much stranger, took its place beside that theory. Whereas until that time electricity had been considered a fluid and had been the very model of an agent with a continuous effect, it too now proved to be made up of particles, namely, positive and negative *electrons*.

Besides matter and electricity there is in physics still something else that is real, for which the law of conservation also holds, namely, energy. Now not even energy, as we know today, permits of infinite division in an absolute and unrestricted way; Planck discovered that energy comes in *quanta*.

And the net result is, certainly, that we do not find anywhere in reality a homogeneous continuum that permits of continued division and hence would realize the infinite in the small. The infinite divisibility of a continuum is an operation that is present only in our thoughts; it is merely an idea, which is refuted by our observations of nature and by the experience gained in physics and chemistry.

We find the second place at which the question of infinity confronts us in nature when we consider the universe as a whole. Here we must investigate the vast expanse of the universe to see whether there is something infinitely large in it.

For a long time the opinion that the world is infinite was dominant; until the time of Kant and even afterward no one had entertained any doubt whatsoever about the infinitude of space.

Here again it is modern science, especially astronomy, that raises this question anew and seeks to decide it, not by the inadequate means of metaphysical speculation, but through reasons that are supported by experience and rest upon the application of the laws of nature. And weighty objections against infinity have appeared. *Euclidean* geometry necessarily leads to the assumption that space is infinite. Now, to be sure, Euclidean geometry, as a structure and a system of notions, is consistent in

itself, but this does not imply that it applies to reality. Whether that is the case, only observation and experience can decide. In the attempt to prove the infinitude of space in a speculative way, moreover, obvious errors were committed. From the fact that outside of a region of space there always is still more space it follows only that space is unbounded but by no means that it is infinite. Unboundedness and finitude, however, do not exclude each other. In the geometry usually referred to as *elliptic*, mathematical research furnishes the natural model of a finite world. And the abandonment of Euclidean geometry is today no longer merely a purely mathematical or philosophical speculation; rather, we have come to abandon it also on account of other considerations, which originally had nothing at all to do with the question of the finitude of the world. Einstein showed that it was necessary to relinquish Euclidean geometry. On the basis of his theory of gravitation he attacks the cosmological questions and shows that a finite world is possible, and all the results discovered by astronomers are compatible also with the assumption of an elliptic world.

We have now ascertained in two directions, toward the infinitely small and toward the infinitely large, that reality is finite. Yet it could very well be the case that the infinite has a well-justified place *in our thinking* and plays the role of an indispensable notion.

(continued on next page.)

Now Cantor, in following these thoughts, developed the theory of transfinite numbers in a most successful way and created a complete calculus for them. So, finally, through the gigantic collaboration of Frege, Dedekind, and Cantor the infinite was enthroned and enjoyed the period of its greatest triumph. In the boldest flight the infinite had reached a dizzy pinnacle of success.

The reaction did not fail to set in; it took very dramatic forms. Events took quite the same turn as in the development of the infinitesimal calculus. In their joy over the new and rich results, mathematicians apparently had not examined critically enough whether the modes of inference employed were admissible; for, purely through the ways in which notions were formed and modes of inference used—ways that in time had become customary—contradictions appeared, sporadically at first, then ever more severely and ominously. They were the paradoxes of set theory, as they are called. In particular, a contradiction discovered by Zermelo and Russell had, when it became known, a downright catastrophic effect in the world of mathematics. Confronted with these paradoxes, Dedekind and Frege actually abandoned their standpoint and quit the field; for a long time Dedekind had reservations about permitting a new edition of his epoch-making booklet (*1888*), and Frege, too, was forced to recognize that the tendency of his book (*1893, 1903*) was mistaken, as he confesses in an appendix. From the most diverse quarters extremely vehement attacks were directed against Cantor's theory itself. The reaction was so violent that the commonest and most fruitful notions and the very simplest and most important modes of inference in mathematics were threatened and their use was to be prohibited. There were, to be sure, defenders of the old; but the defensive measures were rather feeble, and moreover they were not put into effect at the right place in a unified front. Too many remedies were recommended for the paradoxes; the methods of clarification were too checkered.

Let us admit that the situation in which we presently find ourselves with respect to the paradoxes is in the long run intolerable. Just think: in mathematics, this paragon of reliability and truth, the very notions and inferences, as everyone learns, teaches, and uses them, lead to absurdities. And where else would reliability and truth be found if even mathematical thinking fails?

But there is a completely satisfactory way of escaping the paradoxes without committing treason against our science. The considerations that lead us to discover this way and the goals toward which we want to advance are these:

(1) We shall carefully investigate those ways of forming notions and those modes

[1] [[Unlike Cantor, Hilbert takes the number classes to be cumulative; see his definition of the second number class below, p. 386.]]

of inference that are fruitful; we shall nurse them, support them, and make them usable, wherever there is the slightest promise of success. No one shall be able to drive us from the paradise that Cantor created for us.

(2) It is necessary to make inferences everywhere as reliable as they are in ordinary elementary number theory, which no one questions and in which contradictions and paradoxes arise only through our carelessness.

Obviously we shall be able to reach these goals only if we succeed in completely clarifying *the nature of the infinite.*

We saw earlier that the infinite is not to be found anywhere in reality, no matter what experiences and observations or what kind of science we may adduce. Could it be, then, that thinking about objects is so unlike the events involving objects and that it proceeds so differently, so apart from all reality? Is it not clear, rather, that when we believed we had discovered that the infinite was in some sense real we were only allowing ourselves to be led to that belief by the circumstance that we so often actually encounter in reality such immeasurable dimensions in the large and in the small? And has the contentual logical inference ever deceived and abandoned us anywhere when we applied it to real objects or events? No, contentual logical inference is indispensable. It has deceived us only when we accepted arbitrary abstract notions, in particular those under which infinitely many objects are subsumed. What we did, then, was merely to use contentual inference in an illegitimate way; that is, we obviously did not respect necessary conditions for the use of contentual logical inference. And in recognizing that such conditions exist and must be respected we find ourselves in agreement with the philosophers, especially with Kant. Kant already taught—and indeed it is part and parcel of his doctrine—that mathematics has at its disposal a content secured independently of all logic and hence can never be provided with a foundation by means of logic alone; that is why the efforts of Frege and Dedekind were bound to fail. Rather, as a condition for the use of logical inferences and the performance of logical operations, something must already be given to our faculty of representation [[in der Vorstellung]], certain extralogical concrete objects that are intuitively [[anschaulich]] present as immediate experience prior to all thought. If logical inference is to be reliable, it must be possible to survey these objects completely in all their parts, and the fact that they occur, that they differ from one another, and that they follow each other, or are concatenated, is immediately given intuitively, together with the objects, as something that neither can be reduced to anything else nor requires reduction. This is the basic philosophical position that I consider requisite for mathematics and, in general, for all scientific thinking, understanding, and communication.

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From Rebecca Goldstein's Book "Incompleteness"

The Vienna Circle

In addition to café society, the intellectual life of Vienna was also organized into various *Kreise*, or circles, more or less formal discussion groups that met on a weekly basis, centered around the leading intellectuals of the city. Many of these circles overlapped. Some were connected with the university, others not. A large number were devoted to discussions of socialism (one, surrounding Max Adler, was Kant-focused), and others were oriented around the various factions within the psychoanalytic movement. A large number of the circles were meant for the discussion of philosophy, not only of Kant, but of such figures as Kierkegaard and Leo Tolstoy, who enjoyed an enormous influence at the time. The philosopher Heinrich Gomperz, in whose class Gödel had become convinced of Platonism, had a discussion group centered on the history of philosophy. The intellectual geometry of Vienna was densely inscribed with circles.

By far the most prominent of these circles was the one that revolved around the philosopher Moritz Schlick, first dubbed, accordingly, the Schlick Kreis, though it came eventually to be known, as an acknowledgment of its preeminence, as the *Der Wiener-Kreis,* the legendary Vienna Circle. It was from this group of thinkers that the influential movement known as "logical positivism" largely disseminated. The reforming edicts of the group reshaped attitudes of scientists, social scientists, psychologists, and humanists, causing them to reformulate the questions of their respective fields; the effects are still with us.

Attendance at the meetings of the Vienna Circle was by invitation only. The philosopher Karl Popper, who went on to eminence and was even then an up-and-coming intellectual force, waited with impatience and in vain for an invitation to join the most important *Kreis* in town.

Kurt Gödel was invited to join while still an undergraduate and was a regular attendant at the weekly sessions between the years 1926 and 1928. Interestingly, 1928 is the year when he turned to mathematical logic, which would of course yield him his famous proof. No wonder he no longer had the time or the inclination for the weekly sessions.

Gödel had become a Platonist in 1925, a year before joining the discussion group. Their anti-metaphysical orientation had no influence on him, and, for their part, they never seemed to suspect—not for a long time at least—that he was not one of them. He apparently gave them little indication. It was not then, and never would be, in his nature to argue face-to-face with those with whom he disagreed. His distaste for engaging in conflict was so extreme as to qualify as an eccentricity, though hardly among his most pronounced. He refused to oppose another person's viewpoint unless he had absolute certainty on his side, unless, that is, he had a *proof.* All his life, he wanted to have his mathematical proofs do all his speaking for him. (Perhaps it is no accident that this man, whose extreme reticence cloaked intense convictions, should have produced the most prolix mathematical results in the history of mathematics.) He was dismayed when others did not catch all that he was trying to say in them. He was dismayed until the end of his life that people still considered his views consistent with those of the Vienna Circle.[5]

What *were* the views of the Vienna Circle? Logical positivism was first and foremost a movement that spoke in the name of the precision and progress associated with the sciences. It sought to appropriate the methodology that had served the sciences so well, to distill the essence of this methodology not only to cleanse science itself of its more mystically vague and metaphysical tendencies—no characterization carried more positivist opprobrium than "metaphysical"—but also similarly to cleanse all intellectual areas. It was a program for intellectual hygiene.

In the Viennese spirit of the time, this group of thinkers from various fields—mathematics, philosophy, the physical and social sciences—were intent on giving the decaying remains of old ideas as hasty a burial as decency required and on resurrecting in their stead a system whose wholesome soundness would derive from the empirical sciences. Logical positivism disseminated out far beyond the little bare room where the group would meet and deeply penetrated the philosophical orientation of philosophers, scientists, and social scientists, many of whom were not even aware that they *had* a philosophical orientation. But the preferred absence of a specifically philosophical orientation was one of the major points emphasized by the logical positivists. It was a philosophical orientation meant to abolish all philosophical orientations, which might strike the reader as paradoxical.

To use a favorite example, consider the question of the existence of God, defined as a transcendent Being who stands outside space and time, severely limiting the possibilities for experiential contact. (At the least, such experiences would have to occur in time.) Many traditional empiricists had declared the existence of such a trans-empirical God inviolably unknowable, since the cognitive means at our disposal are in principle inadequate for answering the question one way or the other. So remote a God—beyond our experience—may exist, but we'll never know. (Bertrand Russell, when asked what he would say were he to find himself before the pearly gates face-to-face with the Almighty, quipped that his response would be, "Oh Lord, why did you not provide more evidence?")

The logical positivists turned the empiricist theory of knowledge into a theory of meaning. According to the latter, the empirical means that would be relevant to discovering whether a particular proposition is true also provide the very *meaning* of the proposition. The positivist theory of meaning is therefore often called "the verificationist criterion for meaningfulness," and it legislates that the borders of empricial knowability map the borders of meaningfulness. If one cannot, in principle, imagine any possible set of experiences that would count as corroboration for the proposition, then what one has is the mere semblance of a proposition, hollowed out of meaning, what the positivists dubbed a "pseudo-proposition."

By declaring the limits of knowability one and the same with the limits of meaningfulness, the positivists took the problematic aspect of such questions as the existence of God (or of moral values or of abstract entities) up a notch, so that now the unanswerability of certain questions no longer takes the measure of our cognitive inadequacies, but rather signals that the questions ought never have been posed at all. Unknowability is regarded as a sign that a mistake in the use of language has been made. If God (or moral values or universals or numbers) is so defined that no empirical data could possibly be relevant to the question of his (or their) existence, then that question is exposed as ipso facto meaningless: nothing could count as a genuine answer to it.

The positivist transformation of the empiricist theory of knowledge into a theory of meaning meant that the single damning word "meaningless" was to be pronounced over the remains of much that had formerly passed for knowledge. Here was the single word with which to accomplish a program of cognitive hygiene such as the world had never seen. The Vienna Circle, which lasted from 1924 to 1930, ending with the tragic murder of Moritz Schlick by a psychotic former student,[6] had an effect that rippled out from Vienna and is still actively circulating today, quite often in the introductory "philosophical" chapters of textbooks in science or social science.

Dramatis Personae of the Vienna Circle

Moritz Schlick, if not the most dynamic and innovative of the thinkers of the Circle, was a man whose positivist sincerity and organizational abilities seem to have been instrumental to its success. As philosopher Rudolf Carnap said, "The pleasant atmosphere at the meetings of the Circle was due above all to Schlick's personality, his inexhaustible friendliness, tolerance and modesty." Having trained as a physicist in Germany under the great Max Planck, he had come to Vienna in 1922 to take up the prestigious chair in the Philosophy of the Inductive Sciences at the university, the very chair that had been held both by Ernst Mach and by the towering physicist, Ludwig Boltzmann.

Schlick was sympathetic to the drift of the Viennese über-conversation and his presence at the university soon attracted like-minded thinkers from across many disciplines. At first they gathered in an old Vienna café. But the numbers of those participating gradually grew and, in 1924, Schlick agreed to make the gatherings somewhat more formal, moving the group to a room at the university.

Though all (or almost all) in the Circle held positivist views and everyone (even the clandestine Platonist) had either a connection to or a deep sympathy for the exact sciences, there was a diversity of interests and personalities and opinions among them. There was, for example, Rudolf Carnap, who had been trained as a physicist and mathematician at Jena, where he had been influenced by the logician Gottlob Frege (1848–1925). Carnap was "especially interested in the formal-logical problems and techniques," and would have been a happy man indeed to have seen every question reduced to a straightforwardly technical one—the recalcitrantly irreducible of course declared meaningless. He was said to have had a face, especially in his youth, "that almost seemed to exude sincerity and honesty." His intellectual earnestness impressed his fellow positivists; he worked and learned constantly. When anything came up in conversation that was new to him or that he wanted to follow up, he would produce a little notebook and jot down a few words. His ease in writing soon made him the leading exponent of the Circle's ideas.

Otto Neurath was a social scientist and economist, a great big elephant of a man (he signed his letters with a picture of an elephant) with elephantine resources of energy and capacities for enjoying life. Both Carnap (who was an introvert) and Neurath (who was not) had the instinct for political utopianism; and Neurath, in particular, tried to push the Circle in political directions, often making it seem, perhaps unintentionally, that there was more political homogeneity within the Circle than there in fact was. "Schlick especially seemed to resent this since in Vienna, the Circle was named after him, the *Schlick-Kreis*."

Neurath and Carnap felt also that the Circle was intimately connected with other cultural movements, in particular arguing for an affinity between their point of view and the industrial-design-inspired ideology of the Bauhaus. Both were an expression of the *neue Sachlichkeit*, the "fact-mindedness" that received the seal of approval from the sciences. And then in Germany there was the "Berlin Group," centered around the philosopher of science Hans Reichenbach, whose outlook was all but identical with that of the Schlick Circle.

Neurath's sister, the blind, cigar-smoking Olga Neurath, was also an active member of the Circle. She was a mathematician with wide tastes that extended into logic. In her youth she had written three papers, one of which, on the algebra of classes, is described by Clarence I. Lewis in his *Survey of Symbolic Logic* as "among the most important contributions to symbolic logic."

Olga Neurath was married to Hans Hahn, who was also an important member of the Circle. Hahn had been responsible for bringing Schlick from Germany to Vienna. He was a first-rate mathematician, whose name prominently lives on in the useful Hahn-Banach extension theorem in functional analysis. Hahn's mathematical interests were wide, and eventually he became interested in logic. It was he who brought the work in mathematical logic of the German Gottlob Frege and the English Bertrand Russell to the forefront of the Circle's attention. He had an unbounded admiration for Russell and did the Vienna Circle the great service of saving them the difficulty of reading through the monumental three-volume *Principia Mathematica,* explaining it all to them in his seminar of the academic year 1924–25.

Hans Hahn is of particular interest in our story because when Gödel decided to switch his focus from number theory to mathematical logic, Hahn became his dissertation advisor. Though Hahn's specialty was not logic (though he had done some significant work in set theory) his mathematical interests were certainly flexible enough to accommodate Gödel's new interest. Gödel had first come into contact with Hahn in 1925 or 1926, and he told Hao Wang that Hahn had been a first-rate teacher, explaining everything "to the last detail."

Noch Einmal: Man Is the Measure of All Things

In 1929, when Schlick rejected an offer of a prestigious and lucrative professorship in his native Germany, the other members of the Circle decided to celebrate by publishing, in Schlick's honor, a booklet setting out the tenets and aims of their joint point of view. The result was a sort of positivist manifesto entitled *Wissenschaftliche Weltauffassung: Der Wiener Kreis*, or *The Scientific Worldview: The Vienna Circle.* "Everything," it proclaimed, "is accessible to man. Man is the measure of all things." The ancient Sophist's words were reiterated verbatim, only now given a scientifically minded twist: whatever question is, in principle, not susceptible to measurement, that is, empirical procedures, is no question at all. Since the limits of knowability are congruent with the limits of meaning, no meaningful matter can escape our grasp. We are cognitively complete.

Wittgenstein and the Circle

By far the most influential figure connected with the Vienna Circle was not even a member of it, and in fact steadfastly refused membership. This was the philosopher Ludwig Wittgenstein. Wittgenstein, at least according to the interpretation that I will propose, plays a significant, if ambiguous, role in the story of Gödel's incompleteness theorems. Wittgenstein's almost mystical influence on the members of the Vienna Circle, the esteemed thinkers among whom the young logician first came to think rigorously about the foundations of mathematics, must have struck a person of Gödel's persuasion as highly dubious. There are still-smoldering remnants of Gödel's resentment of the philosopher to be found in the *Nachlass,* written (though never exposed to the public) many decades after the Vienna Circle had ceased to be, only a few years before the logician's death.

Gödel's and Wittgenstein's views on the foundations of mathematics were, as we will see, at loggerheads, and neither could acknowledge the work of the other without renouncing what was most central in his own view. Each, I believe, was a thorn deep in the other's metamathematics.

Wittgenstein came from one of the wealthiest and most culturally elite families of Vienna, "the Austrian equivalent of the Krupps, the Carnegies, the Rothschilds, whose lavish palace on Alleegasse had hosted concerts by Brahms and Mahler, Clara Schumann, and the conductor Bruno Walter."[9] He was, in his intensity, preoccupations, ambitions, and conflicts, indelibly stamped by the sensibilities of that intense, preoccupied, ambitious, and conflicted city. While studying aeronautical engineering at the Technische Hochschule in Berlin, he had learned of Russell's paradox, and became interested in the foundations of mathematics.

Wittgenstein went to Cambridge, where Russell was the most prominent philosopher on staff, and immediately made himself known to the distinguished philosopher, mathematician, political activist, and aristocrat.[12]

At first Russell was a bit wary before the strange intensity of the newcomer: "My ferocious German [sic] came and argued at me after my lecture," Russell wrote But within a short span of time (while Wittgenstein was still an undergraduate) the "ferocious" convictions of the Austrian had a devastating effect on Russell's confidence in his own logical powers:

> We were both cross from the heat—I showed him a crucial part of what I had been writing. He said it was all wrong, not realizing the difficulties—that he had tried my view and knew it couldn't work. I couldn't understand his objection—in fact he was very inarticulate—but I feel in my bones that he must be right, and that he has seen something I have missed. If I could see it too I shouldn't mind, but, as it is, it is worrying and has rather destroyed the pleasure in my writing—I can only go on with what I see, and yet I feel it is probably all wrong, and that Wittgenstein will think me a dishonest scoundrel for going on with it.

Back in Vienna, Wittgenstein, in absentia, was also producing a profound effect. His first published work, *Tractatus Logico-Philosophicus*, partly written in the trenches of the First World War, had singularly impressed Schlick's group. As stylistically arresting as its creator, this work achieved in its austere elegance a sort of poetry.[14] The traditional tool of the philosopher—the argument—is dispensed with; each assertion is put forth, as Russell once remarked, "as if it were a Czar's ukase." The poet's obscurity of meaning is preserved despite (by means of?) the formal precision of its elaborate numbering system, which hierarchically arranges its assertions: so that, say, proposition 3.411 (*In geometry and logic alike a place is a possibility: something can exist in it*) is an elaboration of proposition 3.41 (*The propositional sign with logical co-ordinates—that is the logical place)* which is an elaboration of 3.4 (*A proposition determines a place in logical space*). The numbering system is borrowed from the mathematician Peano, who had used it in axiomatizing arithmetic, and it is the numbering system that Russell and Whitehead had also employed in *Principia Mathematica.*

Bertrand Russell wrote the introduction that finally, after much difficulty, secured the author a publisher. Wittgenstein detested the introduction, especially after it was translated into German: "All the refinement of your English style," he wrote Russell, "was, obviously, lost in the translation and what remained was superficiality and misunderstanding." Russell's and Wittgenstein's former intimacy cooled considerably over the following years. "He had the pride of Lucifer," was one of Russell's later summations of Wittgenstein's character.

It was Kurt Reidemeister, a geometer associated with the Circle, who in 1924 or 1925, at Schlick's and Hahn's request, studied the *Tractatus* and suggested that the group read it together.

And so the positivists began a joint study of the *Tractatus,* proposition by proposition, their Thursday-evening meetings now dedicated to Wittgenstein. They read it through not once, but twice, the endeavor taking the better part of a year.

The Viennese positivists interpreted the cryptic *Tractatus* as offering precisely the new, purifying foundations they sought. Proposition 4.003, for example, could not summarize more perfectly their fundamental conviction:

> Most of the propositions and questions to be found in philosophical works are not false but nonsensical. Consequently we cannot give any answer to questions of this kind, but can only point out that they are nonsensical. Most of the propositions and questions of philosophers arise from our failure to understand the logic of our language. . . . And it is not surprising that the deepest problems are in fact not problems at all.

They also believed that Wittgenstein had accounted for the truths of mathematics and logic, reducing them to tautologies, devoid of any descriptive content.

Mathematical propositions, just like the tautologies of logic, do not represent any facts because they are, in a certain sense, merely grammatical. "6.233 The question whether intuition is needed for the solution of mathematical problems must be given by the answer that in this case language itself provides the necessary intuition." (Proposition 6.233 also puts him starkly at odds with Gödel's result, as we will see.) By language itself, Wittgenstein means syntax, the rules that stipulate that which can be said. Mathematics, like logic, is syntactic.

Of What We Cannot Speak

Though Wittgenstein may have believed he had summarily disposed of Russell's paradox, the very problem that had drawn him away from aeronautical engineering and into the world of philosophy of logic and language, the entire *Tractatus* constitutes a self-avowed paradox, as the philosopher himself freely admits. According to its own dictates, its very own propositions are meaningless. Wittgenstein forbade talking about a language within the language. The syntactical nature, whether of logic or of mathematics, cannot really, without violating the syntax of the language, be spoken about, but must rather be shown.

> 6.54 My propositions serve as elucidations in the following way: anyone who understands me eventually recognizes them as nonsensical, when he has used them—as steps—to climb up beyond them. (He must, so to speak, throw away the ladder after he had climbed up it.)

(This last metaphor, for which Wittgenstein is famous, was one that Wittgenstein borrowed from the drama critic/philosopher Fritz Mauthner, of whose *Sprachkritik* Wittgenstein tended to be rather critical in the *Tractatus.* 4.0031: "All philosophy is a 'critique of language' [though not in Mauthner's sense"].)

Wittgenstein's attitude toward the inherent contradiction of the *Tractatus* is perhaps more Zen than positivist. He deemed the contradiction unavoidable. Unlike the scientifically minded philosophers who took him as their inspiration, he was paradox-friendly. Paradox did not, for Wittgenstein, signify that something had gone deeply wrong in the processes of reason, setting off an alarm to send the search party out to find the mistaken hidden assumption. His insouciance in the face of paradox was an aspect of his thinking that it was all but impossible for the very un-Zenlike members of the Vienna Circle to understand.[16]

In his autobiography Carnap recalled how the Vienna Circle had struggled with Wittgenstein's dictum concerning the question of whether "it is possible to speak about linguistic expressions."[17] Carnap asked Wittgenstein for elucidation on this point once too often and was summarily banished forevermore from Wittgenstein's presence.