Philosophy of Mathematics

A twelfth grade one-quarter course for track class (last updated April 2016)

Note: These lesson plans have been adapted in order to include Logicomix as the course's textbook.

Overview. In the middle of the nineteenth century, two subjects that have always been considered center pieces of human thought – science and mathematics – were shaken to their core. In the world of science, it was the emergence of the theories of quantum and relativity that shook the world of physics. While many educated people are at least familiar with the terms "quantum" and "relativity", and the brilliant minds that brought it all about (Einstein, Bohr, Heisenberg, Planck, etc.), the story of the "foundational crisis of mathematics", and its implications to the world of mathematics, is known by very few people. This course tells this great story, and in the process investigates some of the classic questions of the philosophy of math.

Notes:

- This course has been planned for one quarter of the year for track class – about 30 lessons. It is the last math class that our students take in high school.
- I teach this course to all of the twelfth grade class – advanced and regular sections combined.
- This is a course that I do because it (the philosophy of math, in general, and the foundational crisis, in particular) is a passion of mine. It wouldn't work for a teacher that doesn't have passion for this material.
- The main reader for the course is the graphic novel *Logicomix*.
- I also hand out a reader at the beginning of the course, which includes the following essays and articles, much of which is found in James R. Newman's wonderful four-volume anthology of mathematics, titled *The World of Mathematics*. **You can now download this reader from our website.**
  - Excerpts from *A Mathematician's Apology* by G. H. Hardy
  - *Mathematics as an Art* by John Williams Navin Sullivan
  - *Mathematics and the Metaphysicians* by Bertrand Russell
  - *Mathematical Creation* by Henri Poincaré
  - *The Crisis in Intuition* by Hans Hahn
  - *Paradox Lost and Paradox Regained* by Kasner and Newman
  - Excerpts from *The Western Heritage from the Earliest Times to the Present* by Stewart Easton
  - *Mathematical Platonism and its Opposites* by Barry Mazur
  - Excerpts from *What is Mathematics, Really?* by Reuben Hersh
  - *Intuitionistic Reflections on Formalism* by Luitzen Brouwer
  - *On the Infinite* by David Hilbert
  - Excerpts from Rebecca Goldstein's book: *Incompleteness: The Proof and Paradox of Kurt Gödel*
- In the 2nd week, I show the class Simon Singh's documentary of Andrew Wiles' proof of Fermat's Last Theorem. The link is: http://video.google.com/videoplay?docid=8269328330690408516#.

Worksheets: I have designed worksheets for this course. Download them from www.jamieyorkpress.com.

Day #1

- Motivational speech: Why it is a privilege to take this course.
  - Some of you may be thinking that you're done with high school. Or, perhaps, since you've already applied to colleges, that your grade in this course doesn't matter.
  - I had a realization in 9th grade that grades aren't what matters most. You have to work hard to get good grades in high school to get into a good college, and then work hard to get good grades in college to get a good job, and then work hard in your job…all for what?
  - My advice to you: find it within yourself to become motivated by learning – for the sake of becoming an educated person – rather then being motivated by grades. Your education will become more meaningful and enjoyable if you focus on learning rather than grades. And you'll likely end up with better grades as well.
  - I hope that you will feel that this is the best math course you have even taken. It has the possibility of being that if everyone does their job, which means doing the readings and participating in the conversations.
- Discussion: What is mathematics? What do most people think that mathematics is? How is math different from science?
- HW: Read Hardy's essay "A Mathematician's Apology" for tomorrow, and "Mathematics as an Art" by Sullivan for the next class after that.

Day #2

- Discussion (perhaps keep it brief): Hardy's essay (which was written in 1940)
  - The myth is that Mathematicians are dull people.
  - In addition to being an advocate of rigor in mathematical proof, he was an ardent atheist as well as an avid cricket player and fan.
  - Hardy once wrote a postcard to a friend containing the following New Year's resolutions: 1. To prove the Riemann hypothesis 2. To make a brilliant play in a crucial cricket match, 3. To prove the nonexistence of God, 4. To be the first man atop Mount Everest, 5. To be proclaimed the first president of the U.S.S.R., Great Britain, and Germany, and 6. To murder Mussolini.
  - Hardy once told Bertrand Russell "If I could prove by logic that you would die in five minutes, I should be sorry you were going to die, but my sorrow would be very much mitigated by my pleasure in the proof" (Clark 1976; Hoffman 1998, pp. 84-85).
  - GH Hardy (7 February 1877 – 1 December 1947) wrote "A Mathematician's Apology" in 1940 at age 63. He was a professor at Cambridge and Oxford for 35 years. "I hate teaching, but love lecturing."
  - Hardy discovered Ramanujan in 1912 (age 25). Ramanujan died at age 33 from TB.
  - Tell Hardy and Ramanujan story. Hardy's taxi was #1729. It is the first number that can be expressed as the sum of two cubes in two different ways.
  - Hardy suffered from depression, attempted suicide late in his life, and died in 1947.
- Discussion: Is math discovered or created (i.e., invented)?
  - If math is discovered, and it "has always been there", how did it get there? Who created it? God?
  - My thought: For many mathematicians, math is the one place they believe (subconsciously) in God.

Day #3

- Discussion: "Mathematics as an Art" (which was written in 1925) by Sullivan (1886-1937)
  - Sullivan was a journalist and "math enthusiast".
  - Be clear about what is meant by "doing math".
  - How is a mathematician different than a normal artist?
  - Does a poet or sculptor discover or create?
- Math Puzzle (If time allows) Intro tomorrow's math puzzle.

Day #4

- Math Puzzle: Solve a good math puzzle (in groups), which allows the students to experience true problem solving. This should take nearly two days of class. Here is a good one, called *Cut Plane*:

  Ten planes divide space into how many regions? (No two planes may be parallel; any three planes must intersect at exactly one point; no four planes may meet at the same point, etc.)
- Solution: We will first solve a simpler version of the problem, and then see what insight it gives to the originally stated problem.

By reducing the number of planes, we can fairly easily visualize that 0 planes yield 1 region; 1 plane yields 2 regions; 2 planes yield 4 regions; and 3 planes yield 8 regions. One might be inclined at this point to see a pattern and conclude that the next step is 4 planes yield 16 regions, and therefore 10 planes will yield 1024 (2^{10}) regions.

However, on closer inspection, we start with the 8 regions that were produced by 3 planes and see that the 4th plane cuts through all but one of these 8 regions, thereby showing that 4 planes divide space into 15 regions. It is even more difficult to picture how many regions are produced by 5 planes. Certainly, trying to picture how many regions are created by 10 planes would be impossible.

The major insight is to simplify the number of dimensions of the original problem. Therefore, we ask: how many regions on a plane are produced by a certain number of lines? (Again, where all of the lines intersect with one another, but never three lines meeting at the same point.) This is quite manageable, and, as long as we are careful, we can progress quite quickly up to nine lines.

2-D

lines	regions
0	1
1	2
2	4
3	7
4	11
5	16
6	22
7	29
8	37
9	46

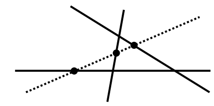
It usually doesn't take long before the students notice the pattern that generates this 2-D table. The differences in the "regions" column keeps going up by one. And once that has been discovered, the students will likely discover two things:

1. The differences in the 3-D table (1, 2, 4, 7, etc.) appear to come from the 2-D table.
2. If we reduce from two dimensions to one dimension
   (i.e., "How many regions on a line are produced by a certain number of points?"), then the resulting

"regions" column is simply the counting numbers (1, 2, 3, 4…). Quite nicely, we see that the differences in the 2-D table are given by the 1-D table, which seems to support our proposition stated in #1, above.

Of course, we can go ahead and fill out the 3-D right away, but we are left with a bit of doubt as to whether proposition #1 (i.e., the differences in the 3-D table come from the 2-D table) is definitely true. How can we demonstrate that this is true?

Here is one possible approach… Let's start with the 2-D case of three lines (which divide the plane into 7 regions). The question is: "How many additional regions are created by the fourth line?"



To answer this question, we start by looking at the drawing on the right. We see that the new 4th line intersects the other lines at 3 points. Each of the 4 segments that the 1-D line is broken into, corresponds to a new region on the 2-D plane.

In other words, our above 2-D question ("How many additional regions are created by adding a 4th line?"), can now be translated into this 1-D question: "How many regions are produced by the 3 points of intersection on this new (4th) line?" The answer to both of these questions is 4, which can be found in the 1-D table, given below.

And in the same way, we can fill in the next place of the 3-D table by transforming this 3-D question: "How many additional regions are created by the 5th plane?", into this 2-D question: "How many regions are produced by the 4 lines of intersection on this new (5th) plane?" The answer to both questions (which is 11) can be found on the 2-D table. We now know 5 planes divide space into 15+11 = 26 regions.

Now, we can fill out our tables, even past three dimensions.

1-D

points	regions
0	1
1	2
2	3
3	4
4	5
5	6
6	7
7	8
8	9
9	10
10	11

2-D

lines	regions
0	1
1	2
2	4
3	7
4	11
5	16
6	22
7	29
8	37
9	46
10	56

3-D

planes	regions
0	1
1	2
2	4
3	8
4	15
5	26
6	42
7	64
8	93
9	130
10	176

4-D

hyper-planes	regions
0	1
1	2
2	4
3	8
4	16
5	31
6	57
7	99
8	163
9	256
10	386

Therefore, the answer to our original question is that 10 planes divide space into 176 regions.

Lastly, we leave with an interesting thought: what would all of this be in projective geometry? Answer: The number of regions would be found one row higher. For example, in 3-D projective space, 7 planes divide space into 42 regions.

- Prepare students for reading Russell's paper.
  - Explain how did we got to the point that axioms were no longer about "unproved self-evident truths", but instead are "unproved meaningless propositions"?
  - He questions some basic commonly held thoughts, such as:
  - That postulates (he calls them propositions) are about truth.
  - He refers to "undefined terms" (point, line, etc.). He says that postulates are now about an accepted, but meaningless, statement. We don't care if it is actually true. Look for what he says math is defined as.
  - The whole is greater than the part. Look for the surprising example of when this isn't true.
- HW: Read Russell's paper: "Mathematics and the Metaphysicians"

Day #5

- Math Puzzle: Finish it, and then ask why we did this? This is real math – real problem solving. How is this different than what you normally think about math?
- Russell's Bio: When telling Russell's bio it is best to use Newman's commentary (World of Math, p377).
  - Full name: Bertrand Arthur William Russell, 3rd Earl Russell.
  - He was a philosopher, historian, mathematician, social reformer, and political activist. He was one of the greatest intellects of the 20th century.
  - Born 1872 into an aristocratic family. His paternal grandfather had been prime minister. His mother was the daughter of a baron Stanley. Both parents died before he turned four.
  - At age five, he was told that the world was round, but refused to believe it because his senses told him otherwise. So he decided to try digging a hole in order to prove to himself whether it was true or not.
  - He was raised by his grandmother and was tutored at home until college.
  - As an adolescent, he was lonely and frequently contemplated suicide. He said that his keenest interests were religion, mathematics, and sex, and that his desire to learn more math kept him from suicide.
  - He was married four times, and had many love affairs.
  - He had several books published, on a variety of topics, including philosophy, politics, economics, physics, mathematics, logic, geometry, education, and religious and moral issues.
  - He taught at the Cambridge (Trinity), but was dismissed from his post because of his outspoken pacifist views, and was imprisoned for six months in 1918 for a pacifistic article he had written. His *Introduction to Mathematical Philosophy* (1919) was written in prison.
  - In 1950 he was awarded the Nobel Prize in Literature for "in recognition of his varied and significant writings in which he champions humanitarian ideals and freedom of thought".
  - He also taught at University of Chicago, and UCLA.
  - Russell was one of the founders of analytic philosophy, and led a revolt against idealism and metaphysics. (Idealism is the philosophical theory which maintains that the ultimate nature of reality is based on mind or ideas. It holds that the so-called external or "real world" is inseparable from mind, consciousness, or perception, as supported in various forms by)
  - Well into his 90's, he campaigned vigorously against the Vietnam war.
  - A quote from Russell: "For four and a half months in 1918, I was in prison for pacifist propaganda; but, by the intervention of Arthur Balfour, I was placed in the first division, so that while in prison I was able to read and write as much as I liked, provided I did no pacifist propaganda. I found prison in many ways quite agreeable. I had no engagements, no difficult decisions to make, no fear of callers, no interruptions to my work. I read enormously; I wrote a book, *Introduction to Mathematical Philosophy*, and began the work for *Analysis of Mind*. I was rather interested in my fellow prisoners, who seemed to me in no way morally inferior to the rest of the population, though they were on the whole slightly below the usual level of intelligence, as was shown by their having been caught…I was much cheered on my arrival by the warder at the gate, who had to take particulars about me. He asked my religion, and I replied 'agnostic.' He asked how to spell it, and remarked with a sigh: 'Well, there are many religions, but I suppose they all worship the same God.' This remark kept me cheerful for about a week."

Day #6

- Discussion: Russell's article (written in 1918):
  - p1577, "Mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true."
  - p1578, "Arithmetic and Algebra require only 3 undefined terms, and 5 postulates."
  - p1578 & 1585: "It is not necessarily true that the whole is greater than the part. This comes up with infinite sets." (This leads to the below intro to Cantor's set theory.)
- Otherwise, this is catch-up day.

Day #7

- Fermat's Last Theorem Documentary: http://video.google.com/videoplay?docid=8269328330690408516#
  - Give a brief explanation of what FLT is, and its history.
  - Watch the first half.
  - Leave 10 minutes at the end for discussion.

Day #8

- Fermat's Last Theorem Documentary:
  - Watch the second half.
  - Leave 20 minutes at the end for discussion.
    - How does this change your idea of what math is, and what it means to be a mathematician?
    - What is the purpose of life?
    - What does it mean to be successful?
- HW: Read the Intro ("Overture") and First Chapter ("Pembroke Lodge") of *Logicomix*: p11-73.

Day #9

- Discussion: Give these questions for *Logicomix* reading, chapter #1
  - What was BR's (Bertrand Russell's) grandmother trying to hide from him?
  - What is the "Quest" that the authors keep referring to?
  - What role did geometric proof (Euclid) play in BR's development?
    - p57: "In Euclid, I found what I had vainly sought in Grandmother's faith."
      "Geometry showed me the only way towards reality: Reason."
      "Physical science derives its power from mathematics."
      "Science is our only hope."
  - What was BR's "terrible disappointment"? Ans: (p70) That with an axiom "even old Euclid has to take something for granted." This then "ignited the rest of my life."
- Lecture: Cantor's Set Theory
  - Cantor said an infinite set is a set that can be put into a one-to-one correspondence with a subset of itself.
  - Some infinite numbers are larger than others. Russell's quote (p1585): "There are infinitely more infinite numbers than finite ones."
  - Example: The orthocenter is always twice as far from the centroid as the circumcenter is from the centroid. Yet at a given instant, they are at the same point equally far away.
    This simply illustrates that $2 \cdot \infty = \infty$.
  - Example: C = set of counting numbers; E = set of even numbers. These two sets can be mapped into a one-to-one correspondence. Therefore they are said to have an equal cardinality – same "number" of members. This is the definition of Countable.
- Give outline of types of numbers. Copy the "Real number Venn Diagram" onto the board. The circles are: Natural, Integer, Rational, Real-Algebraic, Real. The last ring is then Transcendental, and the last two rings is Irrational. (See "Numbers" sheet in notes.)
  - Note: An algebraic number can be created by taking a rational root of a rational number.
- *Perfect Number Puzzle*. A perfect number is a number where the sum of all of its factors (except for itself) is equal to itself. The first three perfect numbers are 6, 28, and 496. Find the next two perfect numbers.
- HW: Read Poincaré article, "Mathematical Creation". (Mention that it gets more interesting the further you read. Ask yourself who he has written the article for? Who is the reader?)

Day #10

- Cantor's Biography (1845-1918):
  - Born in Saint Petersburg, Russia. Moved to Germany at age 11, and then spent the rest of his life there.
  - He was a very accomplished violinist.
  - Cantor spent his entire career as a professor at the University of Halle. Cantor desired a chair at the University of Berlin (a more prestigious university), but Kronecker, who headed mathematics at Berlin until his death in 1891, made it impossible for Cantor to ever leave Halle.
  - Gauss had said that we should "never deal directly with infinity". Cantor was the first to do so.
  - Cantor also proved that the set of points on the real number line (**R**) could be put into a one-to-one correspondence with the set of points in n-dimensional space ($\mathbf{R}^n$). He actually believed that this wasn't possible, and set out to prove it as impossible. This led him to say, "I see it, but I don't believe it!"
  - Cantor's theory of transfinite numbers (first published in 1874) was originally regarded as so counter-intuitive—even shocking—that it encountered resistance from mathematical contemporaries such as Leopold Kronecker and Henri Poincaré and later from Hermann Weyl and L. E. J. Brouwer, while Ludwig Wittgenstein raised philosophical objections.

- Some Christian theologians (particularly neo-Scholastics) saw Cantor's work as a challenge to the uniqueness of the absolute infinity in the nature of God, on one occasion equating the theory of transfinite numbers with pantheism.
- Poincaré referred to Cantor's ideas as a "grave disease" infecting the discipline of mathematics.
- Kronecker's public opposition and personal attacks included describing Cantor as a "scientific charlatan", a "renegade" and a "corrupter of youth."
- Writing decades after Cantor's death, Wittgenstein lamented that mathematics is "ridden through and through with the pernicious idioms of set theory," which he dismissed as "utter nonsense" that is "laughable" and "wrong".
- Cantor's recurring bouts of depression for the last 34 years of his life (starting at age 39) have been blamed on the hostile attitude of many of his contemporaries, but these episodes can now be seen as probable manifestations of a bipolar disorder. He became obsessed with proving that Bacon wrote all of Shakespeare's work.
- Hilbert and Russell gave him much praise. Hilbert said: "No one shall expel us from the Paradise that Cantor has created."

- **Give Cantor's proof that the rational numbers are countable.**
- (optional) Give Cantor's proof that the algebraic numbers are countable. (Needs to be well-prepared!)

Day #11

- Important: Remember what "countable" means. If a set is countable, it is infinitely large, but the same "size" as the set of counting numbers.
- Give Cantor's proof that the real numbers aren't countable! This mean that the real-transcendental numbers aren't countable.

Bio of Henri Poincaré (1854 – 1912)

- He was a French mathematician, theoretical physicist, engineer, and a philosopher of science.
- He is described as "The Last Universalist" since he excelled in all fields of the discipline as it existed during his lifetime.
- He was ambidextrous (wrote equally badly with both hands) and extremely nearsighted. He was physically clumsy and artistically inept.
- He was always in a rush and disliked going back for changes or corrections.
- He never spent a long time on a problem since he believed that the subconscious would continue working on the problem while he consciously worked on another problem.
- Poincaré's work habits have been compared to a bee flying from flower to flower.
- He had an amazing memory for details, words, facts, etc., but was extremely forgetful, often forgetting whether he had already eaten, or not.
- Often, when he did math, he did it all in his head, as he paced around the room furiously muttering to himself. After he had a solution, he would then write it all down.
- Over the course of his mathematical career, Poincare wrote nearly 500 papers on new mathematical ideas.
- As a mathematician and physicist, he made many original fundamental contributions to pure and applied mathematics, mathematical physics, celestial mechanics, and topology. He took some f the first steps in chaos theory, is considered to be one of the founders of topology, and played a role in the formulation of the theory of special relativity.
- He was a leader in the mathematical philosophical school of Intuitionists.
- Poincaré had philosophical views opposite to those of Bertrand Russell and Gottlob Frege, who believed that mathematics was a branch of logic. He vehemently disagreed with the notion that all of mathematics could be rewritten in terms of formal logic. Poincaré claimed that intuition was the life of mathematics.

- Discussion: Poincaré's article (written in 1908).
  - Be sure to outline his 5-step process for mathematical discovery:
    1. Conscious work and effort
    2. Roadblock
    3. Rest/Unconscious work
    4. Sudden illumination
    5. Conscious work

Day #12

The Infinite Hotel. This was first posed by David Hilbert.

- First pose this question: If we multiply infinity by infinity do we get a larger size of infinity? No!

Assume that our hotel has (countably) infinitely many rooms, and the hotel is full.

1. Adding one new guest. Can the hotel accommodate one more guest? Yes! Everyone simply has to move up one room number. Then the new guest takes the fist room.
2. A bus with 1000 new guests. Can the hotel accommodate all of these new guests? Yes! Everyone simply has to move up 1000 room numbers. Then the new guests take the fist 1000 rooms.
3. A bus with (countably) infinitely many new guests. Can the hotel accommodate all of these new guests? Yes! Everyone moves to the room with double the room number. Then the new guests take the odd-numbered rooms.
4. (Countably) infinitely many buses each with (countably) infinitely many new guests. Can the hotel accommodate all of these new guests? Yes! One method is to first have all current guests move to rooms with double their previous room number (as done above). Then label each bus with a prime number, starting at 3. The new guest who was on bus number b, and sitting in seat number n will be assigned to room number b^n.

- HW: Read Chapter #2 of *Logicomix*.

Day #13

Discussion: Give these questions for *Logicomix* reading, chapter #2

- (p87) What does he mean by saying "you are not far off the mark there!"?
- (p94-100) How is this new logic different from logic in the past?
- What role did Whitehead play in BR's life?

- In Class Quick Read: Read Hans Hahn's paper: "Crisis in Intuition"
- Discussion: Compare Hans Hahn's view of intuition with Poincaré's.
  - Hans Hahn (1879-1934) was a very influential mathematician of his time from the Univ. of Vienna. His article "Crisis of Intuition", was published around 1930, and is actually a series of quotes from various lectures he gave in the 1920's.

Day #14

- Hilbert's Talk at the *International Congress of Mathematicians* in Paris in 1900:
  - The general mood was that we had just about reached our goal of finding a new foundational basis for all of mathematics. Hilbert had already constructed a new axiomatic system of Euclidean geometry (using 22 postulates) and proved that the consistency of geometry reduced to the consistency of arithmetic. In 1900, he gave a famous lecture at the International Congress of Mathematicians in Paris. He then presented 23 problems that the world of mathematics should be focusing on in the new century. Among them were:
  - At the same time that math has lost its foundation, so has science (physics) lost its foundation - Newton's Law are no longer unquestionable.
  - What is the common thread? What is really changing? Our thinking!

  (1) Cantor's Continuum Hypothesis – there is no cardinal number between that of the integers (or counting numbers) and that of the real numbers (or irrational numbers).

  (2) A proof that arithmetic (the theory of the natural numbers) is consistent – free of any contradictions.

  (8) Goldbach's Conjecture. (See below)

  (10) Find an algorithm which will determine whether any given Diophantine equation has a solution. A Diophantine equation allows only integral answers. Fermat's Last Theorem is a particular case of a Diophantine problem. Regarding trying to solve Fermat's Last Theorem, Hilbert had said "I should have to put in three years of intensive study, and I haven't that much time to squander on a probable failure." Andrew Wiles finished proving the theorem in 1995, at the age of 42.

- Group exercise: do Worksheet #1 (Download from www.jamieyorkpress.com.)
  - This worksheet leads to the following theorems:
    - Goldbach's Conjecture: "*Every even integer greater than 3 can be written as the sum of two primes.*" This has still not been proven!
    - The Difference of Two Squares: *"Every prime number, except for 2, can be expressed as the difference of two square numbers in one and only one way."*
- HW: Read Newman's paper: "Paradox Lost and Regained"

Day #15

- Lecture: The build up to Russell's paradox.
  - Brief summary of Greek mathematics
  - Separation of math/science from Phil/religion
    - Kepler solves the world's greatest mystery, but believes that the planets are "pushed by angels".
    - LaPlace's quote (which isn't really true) to Napoleon: "I have no need for that hypothesis."
    - How physics had to go through "the eye of the needle".
      - 1800's was the most materialist period – the search for the smallest particle. Suddenly, it all exploded. Newtonian/Cartesian science was no longer infallible. Out of this came relativity and quantum mechanics, which demanded a new kind of thinking.
    - The math world went through a similar crisis (which we will hear about), which few people know about.
  - ca. 1840: The fall of Euclid
  - 1874: Cantor's work and the rise of set theory.
  - 1879-1903: Gottlob Frege's work with logic
    - Frege's life work is provide a logical foundation for arithmetic and build all of mathematics as an extension of logic. For more than 25 years he works towards this goal, publishes a sequence of books along the way: *Begriffsscrift* (1879), *Grundgesetze der Arithmetik, Vol I* (1893), *Grundgesetze der Arithmetik, Vol II* (1903).
    - He develops a whole new language using formal symbols.
    - He wants to show that all of math grows out of logic.
    - Intuition has proved to be unreliable. He wants to remove intuition from any formal math system.
- Finally, in 1902, Frege feels that he has reached his goal with the completion of the second volume of *Grundgesetze der Arithmetik*. But just as it was going to press, Bertrand Russell sends him a paradox, written in a short letter, which destroys Frege's dream. (We will learn more about that paradox tomorrow.)
- Group exercise: Do Worksheet#2 (on the paradoxes) (Download from www.jamieyorkpress.com.)
  - Question #5: S-Sets could either be a member of itself or not. It is consistent, but undecidable.
  - Question #6: R-Sets doesn't work either way. It is inconsistent, self-contradictory, or paradoxical.

Day #16

- Discussion on Newman's paper: "Paradox Lost and Regained"
  - James Roy Newman (1907–1966) was an American mathematician and mathematical historian. In 1956 he published *The World of Mathematics*, a four volume collection of what he felt were the most important essays in mathematics. Newman and Ernest Nagel wrote *Gödel's Proof* in 1958.
- Review: The build up to Russell's paradox.
- (In class) Read the letters sent between Russell and Frege. (Note: a predicate is a mathematical expression, in this case, given in symbolic logic notation.)
  - Frege put an addendum at the end of his *Grundgesetze der Arithmetik* that read as follows:

    > "Hardly anything more unfortunate can befall a scientific writer, than to have one of the foundations of his edifice shaken after the work is finished. I was placed in this position by a letter of Mr. Bertrand Russell, just when the printing of this volume was nearing its completion. The collapse of one of my laws, to which Mr. Russell's paradox leads, seems to undermine not only the foundations of my arithmetic, but the only possible foundations of arithmetic."
- The consequences of Russell's paradox:
  - Frege's system is destroyed. We have no foundation for arithmetic or the whole of mathematics.
  - There is then a huge push to create new system for the foundation of arithmetic and prove that it is both complete and consistent.
    - Complete. A mathematical system is incomplete if there exists a theorem (or statement or formula) that is true but can't be proven within that system.
    - Consistent. A system is inconsistent if two contradictory theorems can be proved true. (e.g., A certain number can be proved as both even and proved as odd.)
  - Three "schools" emerge around this problem, each with a different plan and philosophy.
  - Mention that our focus in the next couple of days will be to discuss these different philosophies of math.
- HW: Read Chapter #3 of *Logicomix*.

Day #17

Quote from Hilbert (from his paper, *On the Infinite*: "Admittedly, the present state of affairs where we run up against the paradoxes is intolerable. Just think, the definitions and deductive methods which everyone learns, teaches, and uses in mathematics, the paragon of truth and certitude, lead to absurdities! If mathematical thinking is defective, where are we to find truth and certitude?"

- Discussion: Give these questions for *Logicomix* reading, chapter #3
  - Give one word to describe Frege.
  - What was Cantor obsessed with?
  - How did BR find out that Gauss was upset at him?
  - According to Poincaré, what does Hilbert think math is?

- Lecture: The 3 Schools
  - The Logistic School:
    - Based upon logic, a continuation, in many ways, of what Frege had started. Russell wanted to "fix" Frege's work by creating a system that makes paradoxes impossible.
    - Early on Russell believes that the axioms of logic are true, later he changes his mind. He believes that "mathematics has no content, merely form" and "the physical meanings we attach to numbers or geometrical concepts are not part of mathematics" (Kline, p1196).
    - Russell and Whitehead published the three volumes of *Principia Mathematica*, in 1910, 1912 and 1913. It was about 2000 pages long. It was never fully completed.
    - *Principia Mathematica*:
      - The number 1 is defined as: $\hat{\alpha}\{\exists x \cdot \alpha = \iota`x\}$ Poincaré said that this definition of 1 was a good definition for those who had never heard of the number 1.
      - The proof that 1+1=2. (on page 379 – shown on the right)

$*54\cdot43.\quad \vdash :. \alpha, \beta \in 1 . \supset : \alpha \cap \beta = \Lambda . \equiv . \alpha \cup \beta \in 2$

Dem.

$\vdash . *54\cdot26 . \supset \vdash :. \alpha = \iota`x . \beta = \iota`y . \supset : \alpha \cup \beta \in 2 . \equiv . x \neq y .$

$[*51\cdot231] \qquad \equiv . \iota`x \cap \iota`y = \Lambda .$

$[*13\cdot12] \qquad \equiv . \alpha \cap \beta = \Lambda \qquad (1)$

$\vdash . (1) . *11\cdot11\cdot35 . \supset$

$\vdash :. (\exists x, y) . \alpha = \iota`x . \beta = \iota`y . \supset : \alpha \cup \beta \in 2 . \equiv . \alpha \cap \beta = \Lambda \qquad (2)$

$\vdash . (2) . *11\cdot54 . *52\cdot1 . \supset \vdash . \text{Prop}$

From this proposition it will follow, when arithmetical addition has been defined, that 1 + 1 = 2.

    - Their publisher couldn't find anyone to evaluate the manuscript, so they figured that if would be willing to be paid to PM, then no one would pay to read it. So Russell had to accept the embarrassment of paying to have the book printed.
    - There is only one person who ever read the book – Kurt Gödel.
  - The Formalist School (Led by David Hilbert)
    - He also wants to provide a foundation for arithmetic and the number system, but he didn't want to use set theory to do it.
    - Hilbert also believed that mathematics was meaningless: "formulas [which consist of meaningless symbols] may imply intuitively meaningful statements, but these implications are not part of mathematics" (Kline p1204).
    - Each branch of math has its own axiomatic foundation. Math is a collection of formal systems.
    - Hilbert developed meta-mathematics hoping to establish the consistency of any formal system.
  - The Intuitionist School:
    - Leopold Kronecker (1823-1891) is the founder. He believed that the whole numbers were given to us by God – all else in mathematics was the creation of man. He objected to the irrational numbers, and said that they were "non-existent", which wasn't upheld by later intuitionists. While alive, Kronecker had no supporters. His ideas gained a following after the paradoxes in 1903.
    - Poincaré was the next important intuitionist. He objected to set theory because it gave rise to the paradoxes. He said that the definition of 1 as given in *Principia* (see above) was a good definition for those who had never heard of the number 1. Poincaré said: "Arithmetic cannot be justified by an axiomatic foundation. Our intuition precedes such a structure."
    - Brouwer believed that mathematical ideas are embedded in the human mind prior to language, logic [which belongs to language], and all experience. Our intuition, not logic or experience, determines the soundness and acceptability of ideas.
    - Hermann Weyl: "Mathematics, nourished by a belief in the absolute that transcends all human possibilities of realization, goes beyond such statements as can claim real meaning and truth founded on evidence."
    - Both the Formalists and the Logicists were trying to save mathematics from the intuitionists.

- HW: Read Brouwer paper. Be aware that Hilbert and Brouwer strongly disagreed!

  Prep for the reading: A debate around the *Law of the Excluded Middle* (that *P* must be either true or false, which is used with an indirect proof) arose, where Brouwer argued that there is a third case: a proposition that is undecidable – neither provable or unprovable. Hilbert responded by saying that "Taking the Principle of the Excluded Middle from the mathematician ... is the same as ... prohibiting the boxer the use of his fists." Hilbert called intutionism a "treason to science".

Day #18

- Discussion on Brouwer paper.
  - Brouwer (1881-1966) was Dutch, a top mathematician and the leading intuitionist during the foundational crisis. This paper was written in 1927.
  - In general, intuitionists allow the use of the law of excluded middle when it is confined to finite sets, but not when it is used in discourse over infinite sets. Brouwer insists that we be more careful about its use.
  - As an example of how ugly the feud between Brouwer and Hilbert got:
    "Mathematische Annalen" was the premier mathematics journal of the time. The most powerful mathematicians sat on its editorial board, including Hilbert, Brouwer, and Einstein. In 1928, Hilbert went through great lengths to get Brouwer kicked off the board, which upset Einstein so much that (I think I read somewhere) Einstein resigned from the board.
  - The intuitionists insisted that math was meaningful, and that its laws were about truth. To the formalists and logicists, math was a "meaningless game". What do you think math is now?
- Lecture on Kant
  - Give general background on the importance of Immanuel Kant (1724-1804).
  - Go over the content found in the document *Notes for Hersh's Essay on Kant*. Hand out these notes to the students afterwards.
  - If time allows, have students start Hersh's essay.
- HW: Read Reuben Hersh's essay on Kant. This is a difficult reading that will likely take at least one hour.

Day #19

- Group exercise: do Worksheet #3 (Download from www.jamieyorkpress.com.)
  - This worksheet leads to the following theorems:
    - Fermat's Little Theorem: (Liebniz proved this in 1683.)
      - Let X be equal to 2. Let Q be any whole number, but don't make it too big. Let $D = x^Q - x$.
        "Whatever X is, if Q is a prime number, then D will always be evenly divisible by Q."
      - It would seem that if Q is composite (i.e., not prime) then D won't be evenly divisible by Q. If X = 2, all of the composite numbers that we could reasonably try would support this statement. But, quite surprisingly, if we let Q=341 (which is evenly divisible by 11), then the resulting 103-digit value for D turns out to be evenly divisible by 341. If X is greater than 2, then generally we can easily find composite values for Q that divide evenly into D. For example, if x = 3 and Q = 6 (composite) produces a D value (726) that is divisible by Q.
    - The Sum of Two Squares:
      - *"If a number is prime and has a remainder of 1 after dividing it by 4, then it can be expressed as the sum of two square numbers in one and only one way."*
      - *"If a number is prime and has a remainder of 3 after dividing it by 4, then it is not possible to express it as a sum of two square numbers."*
      - *"If a number is not prime, then there are a variety of possibilities − it may be that the number can be expressed as the sum of two square numbers in one way, in multiple ways, or not at all."*

Day #20

- Discussion: Hersh's essay on Kant (1724-1804).
  - Essay is from Hersh's book, *What is Mathematics, Really?*, 1997.
  - Have students get into groups and answer the questions from the sheet "Notes for Hersh's Essay", which include the following:
    - What is the dominant philosophy today? Ans: empiricism
    - Was Kant an empiricist or a rationalist? Ans: neither. He was critical of both, and ends up taking certain aspects of both. He believed in a priori knowledge, and in human intuition as a means of gaining knowledge.
    - Classical philosophy begins with Plato and reaches its peak with Kant. Kant's philosophy is a continuation of the Platonic notion of certainty and timelessness in human knowledge. Knowledge can be gained independent of experience, through intuition.

    - All three “schools” had founders (Hilbert as a Formalist, Frege for the logicists, and Brouwer for the intuitionists) that were Kantians.
    - How was mathematical knowledge justified by empiricists? Ans: it had been based upon observations of the physical world with Euclidean geometry.
    - “Even today, people still believe the Euclid myth.” Einstein talks about how non-Euclidean geometry better describes the world of astrophysics.
    - Empiricists then had problems explaining mathematical knowledge, since geometry can no longer be said to be based upon observations of the physical world.
    - The Cheshire cat (from *Alice in Wonderland*). The cat slowly disappears, but his grin remains. This ties into his statement that today’s Platonism wants to give up God, but keep the mathematical ideas created by God.
    - My favorite quote (from p134): Kant: “Two things fill the heart with ever new and ever increasing awe and admiration; the starry heavens above me and the moral laws within me.”
- <u>HW</u>: Read Chapter #4 <u>and</u> the “Entracte” of *Logicomix*.
- <u>Teaser</u>: In 1922 a group of mathematicians and philosophers began to meet weekly in a Vienna café to discuss their philosophical viewpoints. Their philosophy had a profound influence on how our society views math and science, and what the purpose of education is.

Day #21

- <u>Discussion</u>: Give these questions for *Logicomix* reading, chapter #4
    - (p168-171) How did Cantor, Peano, Hilbert and Frege react to BR’s paradox?
    - (p174-175) What was the main purpose of BR’s invention of his “theory of types”?
    - How long did BR and Whitehead work on PM?
    - Why was it so frustrating?
    - What did the publishers think of PM?
    - Who was the only person ever to read the whole thing?
    - What is the main purpose of the *Entracté*? What is the “logic from madness” theme?
- <u>Group Discussion</u>:
    - Leopold Kronecker (intuitionist, 1823-1891) said, “The whole numbers were given to us by God – all else in mathematics was the creation of man.” What do you think of what he said?
    - What are the central questions of philosophy?
      (Answers: What is morality? How do we gain knowledge?
    - Is philosophy important? Is it important how a society views the world?
    - Can a dominant philosophy of the world/life influence our daily lives and politics?
        - Examples: Greece, Rome, Nazis, Communism, Capitalism.
- <u>HW</u>: Read Barry Mazur’s paper on mathematical Platonism.

Day #22

- <u>Lecture</u>: The Vienna Circle and the Logical Positivists
    - In 1922 Moritz Schlick (age 40), who is a prof of phil at Univ of Vienna, and is quite charismatic, organizes a group of mathematicians and philosophers to meet weekly in a Vienna café to discuss their philosophical viewpoints.
    - The group initially includes Rudolf Carnap (age 35), Hans Hahn (age 47, math prof, teacher of Gödel).
    - In 1922, Wittgenstein is 37, Russell is 54, Hilbert 64, Brouwer 45, (Gödel 16).
    - Gödel joins for just two years – from 1926-1928.
    - Ernst Mach arranged mechanics into a deductive system. Mass, length and time are his basic terms. Newton’s laws serve as his axioms. From this all the laws of mechanics are proved. The Vienna Circle declares that this is the model for all of science.
    - The 3 “Bibles”: Mach’s system of mechanics as a model for science; *Principia Mathematica* for mathematics; *Tractatus* (1921) for general philosophy.
    - Vienna Circle “courts” Wittgenstein unsuccessfully. He never attends their actual meetings. He believes that Schlick and others misunderstood *Tractatus*. On occasion, he meets with a few of the Vienna Circle members. In one meeting, Wittgenstein refused to discuss the *Tractatus* at all, and sat with his back to his guests while reading aloud from the poetry of Tagore.

    - Their major stance was that any statement that cannot be supported with empirical evidence is meaningless. E.g., the debate re: parallel lines would be dismissed as meaningless. Therefore, the only purpose of math is as a language to be used in science.
    - They would support Protagoras's (5^{th} century BC) statement "Man is the measure of all things."
    - 1935 Carnap immigrates to U.S., and brings the movement to America (Univ. Chicago & Harvard).
    - 1936 Vienna Circle ends when Schlick is killed by a Nazi student.
- HW: Read excerpt from Rebecca Goldstein's book *Incompleteness* on the Vienna Circle & Logical Positivists.

Day #23

- Discussion on Mazur's paper: "*Mathematical Platonism and its Opposites*"
    - Questions for group discussion:
        - p34: What's the problem with the stealth word, "our"?
        - p34: What are the two main camps regarding "the question"? Led by whom?
        - p35: Is the Platonic view really a theistic stance?
        - p35: For the Platonist, what is the role of proof in mathematics?
        - Is the author a Platonist?
        - Give a summary of the article.
- Bio of Ludwig Wittgenstein (1889-1951)
        - Wittgenstein was born in Vienna in one of the wealthiest families in all of Europe. He had four brothers, three of whom commit suicide – the last of which killed himself when Wittgenstein 30.
        - In 1911, age 22, visits Frege (who was 63). Then went to meet Russell (age 40) in Cambridge – simply walked in on his lecture. Wittgenstein made a great impression on Russell. Studied with Russell for two years.
        - Age 24-27: Retreats (like Descartes) to a remote village in Norway. This is the time that he has most of his ideas for his most important life work – the book *Tractatus Logico Philosophicus*.
        - Age 27-29: Fights for Austro-Hung army in WWI. Writes *Tractatus* while a prisoner in Italy.
        - Regarding *Tractatus*:
            - It is on pure philosophy, written in a similar symbolic/proof style as *Principia Mathematica.*
            - It seeks to identify the relationship between language and reality.
            - Some key ideas (theses) in the book:
                - We can analyse our thoughts and sentences to show their true logical form.
                - Those we cannot so analyze, cannot be meaningfully discussed.
                - Philosophy consists of no more than this form of analysis: ("Whereof one cannot speak, thereof one must be silent").
            - Wittgenstein believes that *Tractatus* resolves for good all problems of philosophy.
        - Age 30-37: He then gave away his entire fortune, abandons his philosophical work, becomes a Christian, and decides to become a school teacher. Tractatus is published (1921). Vienna Circle "courts" Wittgenstein unsuccessfully.
        - Age 38-39: He was physically abusive to his students. He quits as a teacher, goes to live with his sister; depressed; works on her house.
        - Age 40: Returns to Cambridge; has no degree; submits *Tractatus* as his doctoral thesis; Famous quote: slaps Russell on his back and says, "Don't worry, I know you'll never understand it."
        - Age 43: Cuts ties with the Vienna Circle completely.
        - Age 46-49: travels to Soviet Union, Norway, Ireland, and back to Austria (1938) which was quite dangerous.
        - On October 25, 1946, the *Cambridge Moral Sciences Club* had a meeting. Karl Popper (who was 13 years younger than Wittgenstein) presented a paper titled "Are there philosophical problems?", in which he struck up a position against Wittgenstein's, contending that problems in philosophy are real, not just linguistic puzzles as Wittgenstein argued. Wittgenstein was apparently infuriated and started waving a hot poker at Popper, demanding that Popper give him an example of a moral rule. Popper offered one – "Not to threaten visiting speakers with pokers" – at which point Russell had to tell Wittgenstein to put the poker down and then Wittgenstein stormed out. It was the only time the philosophers, three of the most eminent in the world, were ever in the same room together.
        - Remains teaching at Cambridge (head of Phil Dept.) until age 58.
        - Dies in Cambridge at age 62.

Day #24

- Discussion: On Goldstein's essay and on the Vienna Circle & Logical Positivists
    - From Rebecca Goldstein's book: *Incompleteness: The Proof and Paradox of Kurt Gödel*, 2005.
- Review of yesterday.
    - Handout two sheets (downloaded from our website):
        (1) *A Timeline for the Foundational Crisis and the Vienna Circle*.
        (2) *A "Very Brief Summary" of Ludwig Wittgenstein's Tractatus*
    - In groups, have the students read through the propositions from *Tractatus*, and choose their 3 favorites.
- Bio of Kurt Gödel (1906-1978)
    - Kurt Gödel attends Univ of Vienna at age 18. His teacher is Hans Hahn.
    - Soon after, he has attended Hilbert's lectures, and has read *Principia Mathematica*.
    - Age 20, who has recently (and secretly) become a passionate Platonist, starts attending the Vienna Circle meetings. He sits to the side, never says anything, and nobody has a clue that he disagrees strongly with all that is being said.
    - Age 22: He stops attending the Vienna Circle.
    - Gödel's goal: He wants to show that the positivists and formalists are wrong – that math is more than just formal math and has real meaning. His idea is to use PM to show that PM is flawed.
    - Age 25: Publishes his famous Incompleteness Theorems.
    - Age 30 (1936): Schlick is murdered by a Nazi student.
    - In 1940 (age 34), escapes across Russia to Japan in order to immigrate to the U.S. Worked at the Institute for Advanced Studies (Einstein was also there) in Princeton for most of the rest of his life.
    - Died of starvation at age 72 because his wife died and could no longer taste his food (he was paranoid of getting poisoned).
- HW: Read Chapter #5 of *Logicomix*.

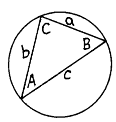
Day #25

- Discussion: Give these questions for *Logicomix* reading, chapter #5
    - How would you describe Wittgenstein?
    - (p237) What is meant by "Europe was sick with nationalism"? What's with the dominoes?
    - (p241-2) What did Wittgenstein learn about language by looking at the military map?
    - Why did Beetle join the army? Why did Wittgenstein join?
    - (p257-261) When BR was reunited with Wittgenstein, what did he think of Wittgenstein's new book (Tractatus) and his new thoughts about philosophy?
- Gödel was a Platonist and believed that statements could be true or false even if they were unprovable.
    - The difference between Platonist and Formalist from the view point of Cantor's Continuum Hypothesis (CH). (From Hersh's "What is Mathematics, Really?", p139):
    - Cantor conjectured that there is no cardinal number between that of the integers (or counting numbers) and that of the real numbers (or irrational numbers).
    - It has been proved that the CH cannot be proven false (1940 by Gödel) or true (1963 by Cohen).
    - Ask the students what the formalist and what the Platonist would say about this?
        - To the Platonist, this means that our system for expressing the CH isn't sufficient to decide. The CH must, of course, either be true or false.
        - The Formalist doesn't agree largely because math isn't about actual truth. Math is about rearranging meaningless symbols (undefined terms). The game of math is about following the rules to create formal proofs. It's pointless to debate whether CH is true or not. Once we knew that it couldn't be proved or disproved, the game was over.
- From Doctor Rick (The Math Forum): "There are two kinds of truth: mathematical truth and real-world truth. Mathematical truth means that a statement is consistent with the assumptions of a particular mathematical system. In a sense, people created that system, and they can tell absolutely whether the statement is true within that system. Real-world truth is of a different order: it means that a statement is consistent with the particular system that is the real world. There is only one real world, and no human created it; no one knows exactly what the rules are. Scientists try to make rules that seem to describe the real world, but they can't possibly know whether these rules really describe everything in the universe." Ask the students which philosophy would agree with this. [Answer: This is really a formalist view. A true formalist believes that the only real math is formal math (i.e., deductive proofs).]

Day #26 **<u>Gödel's proof – Day #1</u>**

- <u>Review</u> (Very briefly – you'll time for the rest of the day's material.)
  - The story of the Vienna Circle.
  - Review the meaning of complete and consistent. The "quest" was to <u>create new system for the foundation of arithmetic and prove that it is both complete and consistent</u>.
  - Russell's goal was to create an axiomatic system (PM) that…
    - Could serve as a new foundation for mathematics. For Russell it was PM, which was based on logic and set theory, written in the language of predicate calculus.
    - avoided paradoxes.
    - was consistent.
    - was complete.
  - Hilbert's goal's were similar to Russell's. Hilbert had already proven that if the consistency of arithmetic was proven, then algebra and (Euclidean) geometry must also be consistent. Hilbert wanted an axiomatic system that included arithmetic, but felt it was critical to <u>prove</u> that the system was both consistent and complete.
  - Gödel's plan/goal:
    - He wants to show that the positivists and formalists are wrong – that math has real meaning and that math is more than just formal math.
    - His idea is to use PM to show that PM is essentially flawed.
    - To create meta-mathematical statements using PM that say something about PM.
    - To create a paradoxical statement using PM's predicate calculus language, even though PM is designed to get around the idea of a paradox.
- Go over the idea of <u>Gödel Numbers</u>.
- A proof is a series of formulas (statements). The last statement of the proof is the formula (theorem) that has been proven. Each formula must be "well-formed"; each step is justified by a rule in the system.
  - <u>Example</u>: A Proof of the Law of Sines (if we have a system that makes that allows this theorem.)

1. Given any ΔABC
2. Circumscribe a circle around it. Let d be the diameter of the circle.



3. $\text{Sin}(A) = \frac{a}{d}$
4. $d = \frac{a}{\sin(A)}$
5. $\text{Sin}(B) = \frac{b}{d}$
6. $d = \frac{b}{\sin(B)}$
7. $\frac{a}{\sin(A)} = \frac{b}{\sin(B)}$
8. $\frac{a}{b} = \frac{\sin A}{\sin B}$

This is a proof of the last statement (which is a consequence of the first statement).

- <u>Do worksheet#4</u>. (Download from www.jamieyorkpress.com.)
- <u>HW</u>: Finish worksheet #4, or at least look it over to refresh your memory.
- In preparation for what is below, go over the idea of a *contrapositive*.
  Use the statement: "If it is snowing, then it is cold."
  - *Converse* (not always true): "If it is cold, then it is snowing."
  - *Contrapositive* (always true): "If it is warm, then it is not snowing."
- Give examples of formulas:
  - $5 = 3 + 2$ (true)
  - $5 \neq 3 + 2$ (not true)
  - $5 = 3 + 1$ (not true)
  - $x = y + 3$ (may be true, depending upon the values of x and y.)

- **<u>Thoroughly go over the first half of the sheet titled "Gödel's Proof – Functions & Formulas"</u>**.
  - Regarding "Key Idea #4": If we could somehow prove something that wasn't actually true, like 2=1, then we could derive lots of other nonsense. In fact, we can find a way to arrive at any (nonsensical) fact we then wish to prove.
      - If 2=1…then
      - 3=2 (by adding 1 to both sides of 2=1)
      - 7=8 (by adding 1 to both sides of 2=1)
      - 6=3 (by multiplying both sides of 2=1 by 3)
      - 10=7 (by adding 4 to both sides of 6=3)
      - Etc.,
  - Regarding Key Idea #5: As stated above, the contrapositive of "If it's snowing, then it's not hot", is "If it's hot, then it's not snowing."
  - Regarding Key Idea #6: *In order to prove that a system is consistent, we only need to be able to find one (presumably false) formula within the system that cannot possibly be proven as true.* For example, if we can prove, by using the axioms of the system, that there is no possible way to prove that 3+9 = 7, then we have proven that the system is consistent.
    <u>Note that proving that 3+9 = 7 is false is very different from proving that it could never be proven as true</u>.
  - Regarding Key Idea #7: Use this example to demonstrate the idea:
    Show the proof that 2=1 (see below), but leave out the conclusion. Ask: "What do you think about this proof?" (They'll not say much.) Then add the conclusion that 2 = 1. Not what do you think? We now know that there must be something wrong with the proof. Why? Because we know that 2=1 is false. This is an example of the following: *Given the statement: "If E is true then F must be true", if we happen to know that F is false, then we can conclude that E cannot be true.*

    <u>Proof that 2=1</u>:
    Let x = y
    $x^2 = xy$
    $x^2 – y^2 = xy – y^2$
    (x+y)(x–y) = y(x–y)
    x+y = y
    2y = y
    2 = 1

Day #27 **<u>Gödel's proof – Day #2</u>**

- Review yesterday.
  - Briefly review the idea of Gödel numbers.
  - Regarding how a proof may look in PM:
    - A proof is a sequence of statements, whereby the whole proof proves the final statement, e.g.:

      Q = ss0 + R
      R = ssss0
      Q = ss0 + ssss0
      Q = ssssss0
    - Each statement can be coded as a single Gödel number.
    - An entire proof can be coded as a single Gödel number.
  - Thoroughly review the *Key Ideas* from "Gödel's Proof – Functions & Formulas"
- You don't have to fully understand what we did yesterday in order to follow today.
- I am only giving an outline of Gödel's proof over these three days – but you should walk away having a good sense of the key ideas behind the proof.
- In order to prove that PM is wrong and that his philosophy of mathematics (Platonism) is right, Gödel decides that he will create meta-mathematical statements within PM. He uses Gödel numbers to do this.
- Announce: Please only ask clarifying questions. It could be extremely difficult to get back "on track" if we get off track.
- Explain the idea of a function. Examples:
  - Is-Even (N). Returns a value of true or false. Is-Even (274) = true
  - Mult (x,y). returns the product of x and y. Mult (4,9) = 36
- **<u>Thoroughly go over the second half of the sheet titled "Gödel's Proof – Functions & Formulas"</u>**.
  - This includes the functions (Dem and Sub) and the formulas (A and G).
  - Formula G is the most important thing to get to.
  - Be sure that notes are on the side chalkboard, and it is saved for tomorrow's review.
- <u>Hanging Question</u>: How is it possible for Gödel to create a statement within PM that he knows is true, but can't be proven within PM?

Day #28 **<u>Gödel's proof – Day #3</u> (The Big Day!)**

- Review the following. (Spend no more than 10 minutes on this.)
  - <u>The Quest</u>. The world of mathematics has been on a quest for 2300 years to find the perfect axiomatic system that can serve as the basis for absolute, certain truth, and the foundation for all mathematical thought. *What would make such an axiomatic system perfect? Answer: If it can be <u>proven</u> to be consistent and complete!*
  - <u>Russell's Goals</u>. To avoid paradoxes and create a system that can serve as the new foundation of mathematics – a system that is both consistent and complete.
  - <u>Godel's Goals</u>. He wants to show that the positivists and formalists are wrong – that math has real meaning and that math is more than just formal math. His idea is to use PM to show that PM is essentially flawed.
  - <u>Review the sheet</u> "Gödel's Proof – Functions & Formulas

- Regarding "Negation". Cover this concept:
  - If C says $x = 7$, then ~C says $x \neq 7$
    If D says "I am happy", then ~D says "I am not happy."
    If D says "I am not tall", then ~D says "I am tall."
    If G says "G is not provable", then ~G says "G is provable."

- **Cover the final sheet of Gödel's proof, "The Central Argument for Gödel's proof".**
  - Before class begins, have all of the above notes on the board. Also write the first assumption on the board, and have at the other end of the board the conclusions written (with the idea that the sentence will become completed at the end). This emphasizes that we are truly beginning with the assumption that PM is consistent. Therefore his conclusions state what must be true <u>if</u> PM is consistent.
    - <u>Conclusion #1</u>: *If PM is consistent, then it is incomplete.*
    - <u>Conclusion #2</u>: *If PM is consistent, then the consistency cannot be proven.*
  - Start class immediately on time. **<u>Allow for at least 30 minutes to cover this new material!!</u>**
  - Start by saying the following:
    - In order to understand today's proof, you don't really have to fully understand the symbols and logic behind formulas A and G. You just have to know their meta-mathematical meaning.
    - Nobody was thinking that arithmetic was inconsistent. But some people (Hilbert in particular) wanted to prove that arithmetic is consistent.
  - End with this statement: **"The quest is impossible!!!"**
  - And then give these *Hanging Questions*:
    - Gödel's proof ended the quest to find the perfect mathematical foundation. What are the implications of Gödel's proof? What did Gödel think of his proof? What did the Vienna Circle think?
- <u>HW</u>: Read Chapter #6 of *Logicomix*.

Day #29 **Important Note:** These next two days are needed so that the students aren't left feeling that Godel's proof is something negative and/or depressing. It takes some real discussion to come to a place of understanding of what the implications really are.

- Discussion:
  - Go over the hanging questions from yesterday:
  - Gödel's proof ended the race to find the perfect mathematical foundation. What are the implications of Gödel's proof? Taken from the footnotes of the "Central Argument" sheet:
    - Gödel also proved that PM is *essentially* incomplete, which means that even if the system is "repaired" by adding more axioms so that it can handle any problematic formulas (like G), then another true, but unprovable formula can always be constructed.
      <u>This destroyed the dream of Russell and PM</u>.
    - And Gödel proves that this is true of *any* formal, axiomatic system that encompasses arithmetic.
      <u>This destroyed the objectives of Hilbert's Program.</u>
- Math, as we knew it, has changed. It demands that we think of math differently.
- Hilbert (and Russell) wanted to give a firm foundation for all mathematics by eliminating intuition from mathematics. Gödel showed that math cannot proceed without intuition.
- Does Gödel's proof say that we can't prove anything to be true?
  - No, we can still prove things to be true.
  - His proof says that we can't have a perfect axiomatic system, nor a perfect foundation for mathematics.
  - "Truth" has not changed. But it seems to be that intuition has to play a part in truth.
- Goldstein's book, *Incompleteness*:
  - "It is extraordinary that a mathematical result can say anything about the nature of mathematical truth in general (meta-mathematics)."
  - "[Gödel believed] that mathematical reality must exceed all formal attempts to contain it."
  - The formalists and positivists were saying that math is a meaningless formal game, reduced to "manipulating meaningless symbols" according to the rules of the game. This game could be played by a machine. Gödel's theorems seem to imply that our minds are not just machines.
- *Important!!* **<u>This proof doesn't mean that math is any less true than before Gödel.</u>** It just means that our traditional means for proving things (i.e., axiomatic systems) are not as flawless as we thought.
- Gödel's proof doesn't say that we can't prove anything (e.g., 3+5 = 8). Of course, arithmetic <u>is</u> true. It does say that we can't prove that an axiomatic system (which includes arithmetic) is consistent.
- The sad irony is that even though Gödel had accomplished what his dream was, he was misunderstood.
  - The logical positivists saw it as evidence of the meaninglessness of math.

    "Some thinkers despaired at this result. Others could never accept it. And still others misunderstood it as a torpedo to the hull of rationality itself. For Gödel, however, it was evidence of an eternal, objective truth, independent of human thought, that can only be imperfectly apprehended by the human mind."

    --from Rebecca Goldstein's book *Incompleteness, The Proof and Paradox of Kurt Gödel.*
- "We know that God exists because mathematics is consistent and we know that the devil exists because we cannot prove the consistency." -- Andre Weil
- As humanity was cutoff more and more from the spiritual world, math education became less philosophical & more mechanical.
- The Logical Positivist's (and Logistic School's) central theme that "mathematics is meaningless" is still, in many ways prevalent today, and has had a big impact on math education.
- <u>From my HS Sourcebook</u>: Many people misunderstood the implications of Gödel's proof. Gödel "defeated" Russell and Hilbert, but sadly, the logical positivists (Vienna Circle) saw it as further evidence of the meaninglessness of math, and this attitude continues to have an impact on mathematics education today.

After Sputnik (the Soviets' first satellite in 1957), America became focused on beating the Soviets. Through this narrow lens, the purpose of mathematics education became to produce more scientists in an effort to win this mad race. Over the past few decades, the world has changed in many ways – socially, economically, and politically. Perhaps now more than ever, there is a realization that mathematics education needs to be overhauled. Opinions differ about where to place the blame, and how to fix it. New mathematics curricula have come and gone. Math textbooks are thicker, "prettier" (more graphics and fancier "packaging"), and more expensive – but less effective than ever. The trends are clear: an increase in teaching to standardized tests; more reliance on technology-driven learning; and teaching more advanced material at a younger and younger age.

And the results? A superficial understanding of material; a loss of a "sense of number"; an inability to do simple calculations in your head; no time left for depth, discovery, or true problem solving. In short, mathematics has become meaningless for most students.

But we must not give up. We all need to become math missionaries. We need to embark on a crusade to make math meaningful.

Day #30 This day is optional

- Possible Questions: With each question, have students talk about it in groups, and then share thoughts with the whole class.
  - What is your view of math now?
  - Is this course worth teaching? (Your HS math education could end with something else.)
  - Now that you are at the end of your HS math education, what advice do you have for math teachers (grades 1-12) about what teachers should consider when teaching math?

(Possible) HW: Paper to write: 1500 words.

- I want to know what your thoughts are.
- It can be on any of the following topics:
  - A summary and/or explanation of Gödel's proof.
  - A topic of your choice related to the Philosophy of Math.

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--- Extra Material (if time allows) ---

- HW: Read excerpts from Stewart Easton's book *Western Heritage*.
- Discuss (briefly) Stewart Easton's reading on *Western Heritage* (written 1966).
  - Group exercise: Have the students answer the following:
    - How would you characterize Plato? Aristotle?
    - What is an idealist?
  - Pythagoras: first philosopher, and number mysticism
  - The Sophists: "The Sophists were the precursors of a more individualistic approach. They were ethical and moral relativists who undermined the primacy of the city-state and called into question all traditions (as Socrates did). I call them the original 'wise guys.' They hired themselves as tutors and teachers to the elite in Athens to teach those aspiring young people how to argue points and be persuasive. But they argued any position without any conviction or sense of what was right or wrong, only how to form an effective argument and rebuttal." ~ Thom Schaefer
  - Plato: Reality starts with his world of Ideas. The particulars come from that.
    Tools of analysis: (spiritual) intuition, logic, imagination
  - Aristotle: Reality starts with observation of the physical world. From observation, we come up with our general ideas. Tools of analysis: empirical observation, logic, systematic thinking.
    - It's not that Aristotle was wrong; it's more that Galileo built upon Aristotle's foundation.
  - Plato and Aristotle were more alike than different. They are often (incorrectly) said to be opposites.
  - Also see Thom Schaefer's sheet "The Dual Legacy".

- HW: Read Hilbert paper, *On The Infinite*. Be aware that Hilbert and Brouwer strongly disagreed!
  - Note: The term "analysis" refers to "calculus".
- Discussion on Hilbert paper (written in 1925, two years before Brouwer's paper).
  - What are the differences between formalism and logicism?
    - From Hilbert's essay: "Kant already taught us that mathematics has…a content secured independently of all logic, and hence can never be provided with a foundation of logic alone."
      He does not believe that PM can be successful.
    - Logicism is the view that mathematics is reducible to logic; all mathematical propositions can be expressed in purely logical vocabulary, and they are all logically true or logically false.
    - Hilbert's "program" was intended to create a new foundation of all of mathematics, and to prove it to be complete and consistent.