$\frac{265}{153} < \sqrt{3} < \frac{1351}{780}$ Goal: Find OA:AG

good to 5 digits

Then $OA : AC [= \sqrt{3} : 1] > 265 : 153$ .... (1),

and 6-gon $OC : CA [= 2 : 1] = 306 : 153$ .... (2).

First, draw OD bisecting the angle AOC and meeting AC in D.

12-gon

Now $CO : OA = CD : DA$, [Eucl. VI. 3]

so that $[CO + OA : OA = CA : DA$, or] CD+OA [Eucl. V-18]

$CO + OA : CA = OA : AD$. [Eucl. V-16]

Therefore [by (1) and (2)]

$OA : AD > 571 : 153$ .... (3).

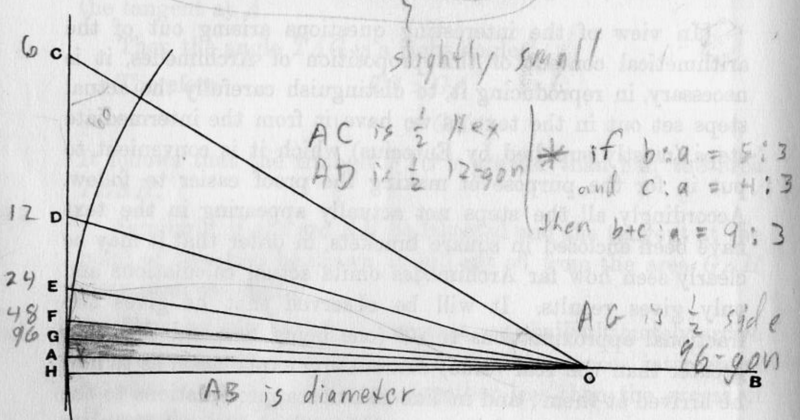
Hence $OD^2 : AD^2 [= (OA^2 + AD^2) : AD^2$

$> (571^2 + 153^2) : 153^2]$

$> 349450 : 23409$, $\frac{1}{7}$ is better

so that $OD : DA > 591\frac{1}{8} : 153$ .... (4).

slightly small



24-gon

Secondly, let OE bisect the angle AOD, meeting AD in E.

[Then $DO : OA = DE : EA$, DO+OA : OA = DE+EA : EA (DA)

so that $DO + OA : DA = OA : AE$.]

Therefore $OA : AE [> (591\frac{1}{8} + 571) : 153$, by (3) and (4)]

OA:AE $> 1162\frac{1}{8} : 153$ .... (5).

24-gon

[It follows that

$OE^2 : EA^2 > \{(1162\frac{1}{8})^2 + 153^2\} : 153^2$

$> (1350534\frac{33}{64} + 23409) : 23409$

$> 1373943\frac{33}{64} : 23409$.] $\frac{1}{7}$ is better

Thus $OE : EA > 1172\frac{1}{8} : 153$ .... (6).

Thirdly, let OF bisect the angle AOE and meet AE in F.

48-gon

We thus obtain the result [corresponding to (3) and (5) above] that

$OA : AF [> (1162\frac{1}{8} + 1172\frac{1}{8}) : 153]$

OA:AF $> 2334\frac{1}{4} : 153$ .... (7).

[Therefore $OF^2 : FA^2 > \{(2334\frac{1}{4})^2 + 153^2\} : 153^2$

$> 5472132\frac{1}{16} : 23409$.]

Thus $OF : FA > 2339\frac{1}{4} : 153$ .... (8).

Fourthly, let OG bisect the angle AOF, meeting AF in G.

We have then

96-gon

$OA : AG [> (2334\frac{1}{4} + 2339\frac{1}{4}) : 153$, by means of (7) and (8)]

OA:AG $> 4673\frac{1}{2} : 153$. our Goal!

Now the angle AOC, which is one-third of a right angle, has been bisected four times, and it follows that

$$\angle AOG = \tfrac{1}{48} \text{ (a right angle)}.$$

Make the angle AOH on the other side of OA equal to the angle AOG, and let GA produced meet OH in H.

Then $\angle GOH = \frac{1}{24}$ (a right angle).

Thus GH is one side of a regular polygon of 96 sides circumscribed to the given circle.

And, since $OA : AG > 4673\frac{1}{2} : 153$,

while $AB = 2OA$, $GH = 2AG$,

it follows that

AB : (perimeter of polygon of 96 sides) $[> 4673\frac{1}{2} : 153 \times 96]$

$> 4673\frac{1}{2} : 14688$.

 The Best approx. with a denom < 4677

But $\frac{14688}{4673\frac{1}{2}} = 3 + \frac{667\frac{1}{2}}{4673\frac{1}{2}}$

$\left[< 3 + \frac{667\frac{1}{2}}{4672\frac{1}{2}}\right]$

$< 3\frac{1}{7}$.

Therefore the circumference of the circle (being less than the perimeter of the polygon) is *a fortiori* less than $3\frac{1}{7}$ times the diameter AB.

12-gon

II. Next let AB be the diameter of a circle, and let AC, meeting the circle in C, make the angle CAB equal to one-third of a right angle. Join BC.

Then $AC : CB\,[= \sqrt{3} : 1] < 1351 : 780$. good to 6 digits

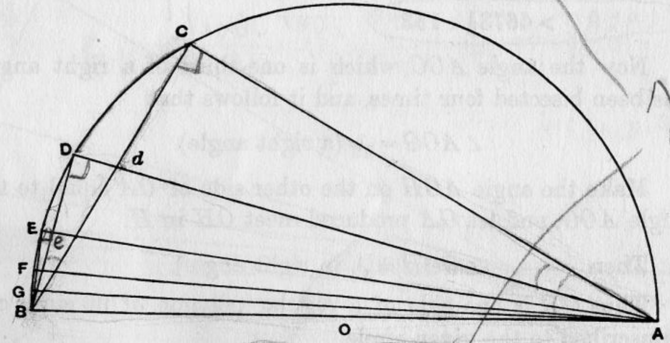
First, let AD bisect the angle BAC and meet BC in d and the circle in D. Join BD.

Then $\angle BAD = \angle dAC$

$= \angle dBD$,

and the angles at D, C are both right angles.

It follows that the triangles ADB, $[ACd]$, BDd are similar. Not used.



Therefore $AD : DB = BD : Dd$

$[= AC : Cd]$

$= AB : Bd$ [Eucl. VI. 3] (V-12)

$= AB + AC : Bd + Cd$

$= AB + AC : BC$

or $BA + AC : BC = AD : DB$.

12-gon

[But $AC : CB < 1351 : 780$, from above,

while $BA : BC = 2 : 1$

$= 1560 : 780.]$ see * p 94

Therefore $AD : DB < 2911 : 780$..................(1).

[Hence $AB^2 : BD^2 < (2911^2 + 780^2) : 780^2$

$AB^2 = AD^2 + BD^2$

$< 9082321 : 608400.]$

Thus $AB : BD < 3013\frac{3}{4} : 780$..............(2).

24-gon

Secondly, let AE bisect the angle BAD, meeting the circle in E; and let BE be joined.

Then we prove, in the same way as before, that

$AE : EB\,[= BA + AD : BD$

$< (3013\frac{3}{4} + 2911) : 780$, by (1) and (2)]

$< 5924\frac{3}{4} : 780$

$< 5924\frac{3}{4} \times \frac{4}{13} : 780 \times \frac{4}{13}$

$AE : EB < 1823 : 240$..........................(3).

[Hence $AB^2 : BE^2 < (1823^2 + 240^2) : 240^2$

$< 3380929 : 57600.]$

Therefore $AB : BE < 1838\frac{9}{11} : 240$..................(4). $\frac{3}{4}$ is better!

48-gon

Thirdly, let AF bisect the angle BAE, meeting the circle in F.

Thus $AF : FB\,[= BA + AE : BE$

$< 3661\frac{9}{11} : 240$, by (3) and (4)]

$< 3661\frac{9}{11} \times \frac{11}{40} : 240 \times \frac{11}{40}$

$AF : FB < 1007 : 66$..........................(5).

[It follows that

$AB^2 : BF^2 < (1007^2 + 66^2) : 66^2$

$< 1018405 : 4356.]$

Therefore $AB : BF < 1009\frac{1}{6} : 66$..................(6).

96-gon

Fourthly, let the angle BAF be bisected by AG meeting the circle in G.

Then $AG : GB\,[= BA + AF : BF]$

$AG : GB < 2016\frac{1}{6} : 66$, by (5) and (6).

96-gon

[And $\quad AB^2 : BG^2 < \{(2016\tfrac{1}{6})^2 + 66^2\} : 66^2$

$$< 4069284\tfrac{1}{36} : 4356.]$$

Therefore $\quad AB : BG < 2017\tfrac{1}{4} : 66,$

whence $\quad BG : AB > 66 : 2017\tfrac{1}{4}$ ....(7).

[Now the angle BAG which is the result of the fourth bisection of the angle BAC, or of one-third of a right angle, is equal to one-fortyeighth of a right angle.

Thus the angle subtended by BG at the centre is

$$\tfrac{1}{24} \text{ (a right angle).]}$$

Therefore BG is a side of a regular inscribed polygon of 96 sides.

It follows from (7) that

$$\text{(perimeter of polygon)} : AB\ [> 96 \times 66 : 2017\tfrac{1}{4}]$$
$$> 6336 : 2017\tfrac{1}{4}.$$

And $\quad \dfrac{6336}{2017\tfrac{1}{4}} > 3\tfrac{10}{71}.$ The best approx with a denom < 220

Much more then is the circumference of the circle greater than $3\tfrac{10}{71}$ times the diameter.

Thus the ratio of the circumference to the diameter

$$< 3\tfrac{1}{7} \text{ but } > 3\tfrac{10}{71}.$$