Cartesian Geometry – Part I

Important Note: This exact same unit also appears in our 11th/12th grade workbook, because in an ideal world, I believe it is best to delay the introduction to Cartesian Geomerty until 11th grade. However, I have made this available here for students who are enrolling in a school where it is expected that students have had at least an introduction to this topic before beginning a mainstream *Algebra II* course.

Problem Set #1

In 1637, René Descartes published a book with the impressive title *Discourse on the Method of Rightly Conducting One's Reason and Searching for the Truth in the Sciences*, which today is seen as a work of major importance in the fields of philosophy and in general science. The book also included an appendix on geometry where he showed how his new method of conducting science could be used to develop a new way of solving geometric problems. Before Descartes, geometry and algebra were separate subjects.

What Descartes actually did was to take a geometry problem (the Pappus problem) and expressed it as an equation. Descartes' seed idea (and Fermat came up with similar ideas at about the same time) was then further developed over a period of time, and has been tremendously influential in the world of science and mathematics. Modern Cartesian geometry (which is also referred to as coordinate geometry or analytical geometry) allows us to take an equation and express it as geometry; *it allows us to visualize algebra*.

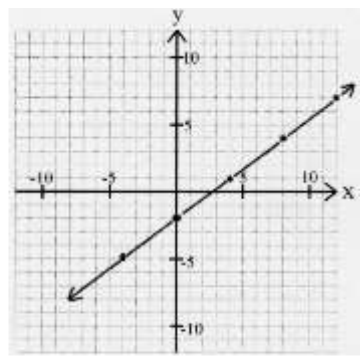
To graph an equation, we simply follow Descartes' words: "We may give any value we please to either x or y, and find the value of the other from the equation", in order to find several solutions. Then you plot the solutions to each equation on a Cartesian graph. For many equations, you have to carefully choose the values for x or y so that the points you plot fit reasonably on the graph.

Example: $y = \frac{3}{4}x - 2$

Solution: It is easiest, perhaps, to plug in multiples of 4 into x. (Although we could have instead chosen fairly random values for x.) This leads to the following table:

x	–4	0	4	8	12
y	–5	–2	1	4	7

Note that for every step of the four that the x values take, the y values take a step of three. (We will see later why this is important.) Lastly, we simply need to graph the solutions that we have found, and we see that the result is the straight line shown here:



Graph each equation.

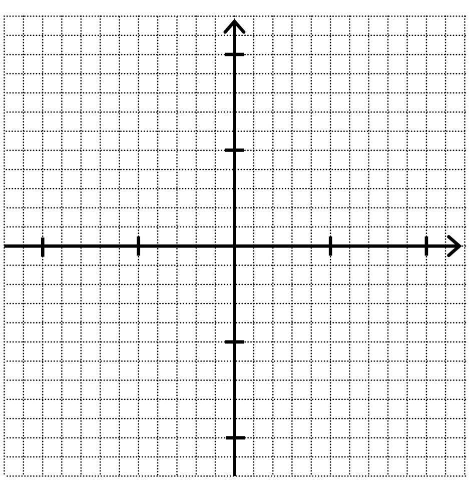
1) $x^2 + y^2 = 25$

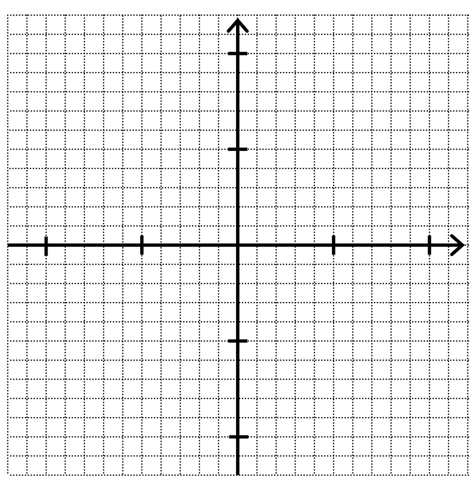
2) $y = \frac{1}{2}x + 3$

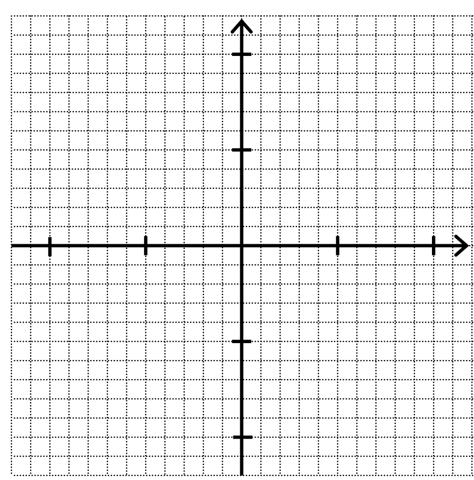
3) $y = x^2 + 6x + 5$

4) $y = 2x^3 - 3x + 1$

5) $y = x^4 - 5x^2 + 4$







Problem Set #2

On the previous problem set we were able to graph equations making a table and then plotting points. While the method of making a table and plotting points can be reliable, it is time consuming and tedious.

You may have noticed on the last problem set that the equations without any exponents ended up having graphs that were straight lines. Such equations are called *linear equations*. Mathematicians are always searching for more efficient ways to do things; this problem set is focused on finding quicker ways to graph linear equations.

1) Graph each of the following on the same graph by making a table and then plotting points.

   a) $y = 2x + 1$

   b) $y = 2x - 3$

   c) $y = 2x + 4$

   d) $y = 2x - 6$

2) With the above equations, what does the number at the end of the equation tell you?

3) Graph each of the following on the same graph by making a table and then plotting points.

   a) $y = 2x + 1$

   b) $y = {}^{2}/_{5}\, x + 1$

   c) $y = {}^{5}/_{2}\, x + 1$

   d) $y = -{}^{5}/_{2}\, x + 1$

   e) $y = -{}^{3}/_{5}\, x + 1$

4) In each of the above equations, what does the number before the "x" tell you?

5) Now, given what you have learned above, graph each of the following <u>without</u> making a table .

   a) $y = {}^{3}/_{2}\, x - 4$

   b) $y = -\,{}^{1}/_{3}\, x - 2$

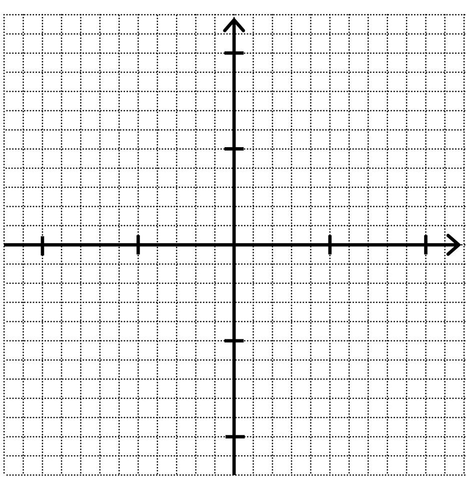
   c) $y = -\,3x$

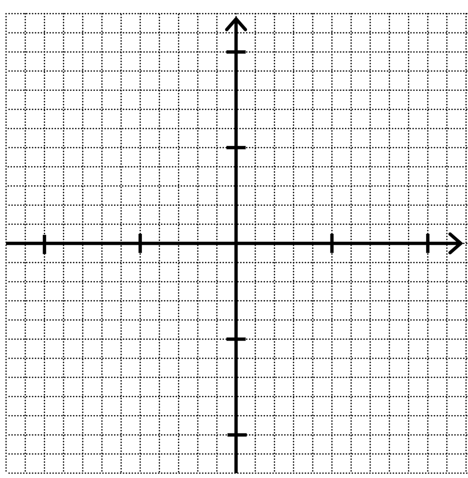
Two Forms

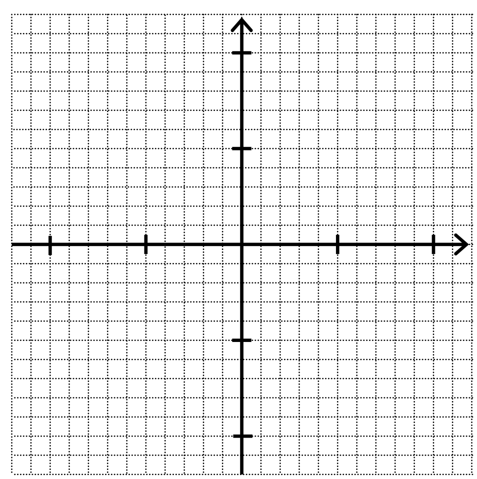
In general, there are two common forms for expressing linear equations. One is called *standard form*, where there are no fractions and the x's and y's are both on the left side, such as:

$$4x + 3y = 15$$

If we now solve this equation for y, then we get *slope-intercept form*, which for the above equation is: $y = -\frac{4}{3}x + 5$



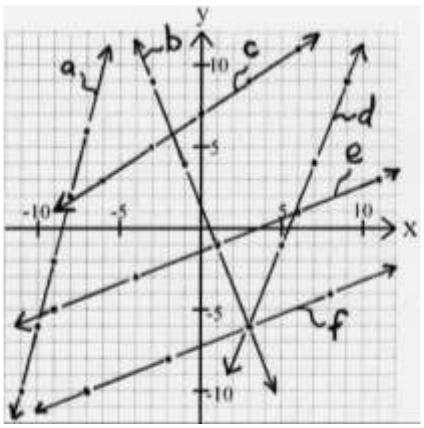




Problem Set #3

On the previous problem we saw how if we have an equation solved for y, then the number in front of the x (which is called the *slope*) tells us how steep the line is, and the constant at the end tells us where the line crosses the y-axis (and this is called the *y-intercept*, and it is where the value for x is equal to zero).

1) Give the slope of all of the lines below.



Questions about slope…

(It may be helpful to look at your answers to the previous problem.)

2) What does a negative or positive slope tell us about the direction of the line?

3) What is the slope of a line that is 45° (off horizontal)?

4) What can be said about the slope of a line that is less steep than 45°?

5) What can be said about the slope of a line that is steeper than 45°?

6) What can be said about the slopes of two lines that are parallel with each other?

7) What can be said about the slopes of two lines that are perpendicular to each other?

8) What is the slope of a line that is horizontal?

9) What is the slope of a line that is vertical?

10) Graph the following equations.

a) $y = \frac{1}{3}x - 4$

b) $y = \frac{3}{2}x - 1$

c) $y = -\frac{3}{2}x + 1$

d) $y = \frac{3}{4}x + 2$

e) $y = \frac{1}{2}x$

f) $y = -5x$

g) $y = 2x - 5$

h) $y = -3x + 2$

i) $y + 3x = 2$

j) $4y - x = -8$

k) $y = 4$

l) $3x + 2y = 2$

11) The last equation above is the same as what other equation given further above?

12) Consider the equation

$y = x^2 - 4x$.

a) Give three solutions to the equation.

b) What are all the possible values that x can have? (In other words, is there a limit to how big or small x can be?)

c) What are all the possible values that y can have? (In other words, is there a limit to how big or small y can be?)

d) Graph the equation.

e) Does graphing the equation give you any insights into the answers to part b and c?

13) Consider the equations

$4y + 3x = 6$

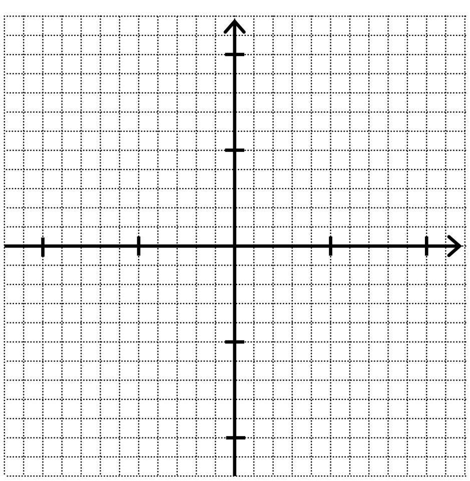
and $y - 2x = 7$

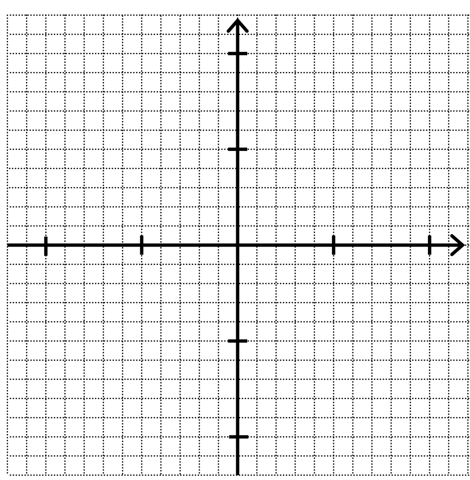
a) Give three solutions to each equation.

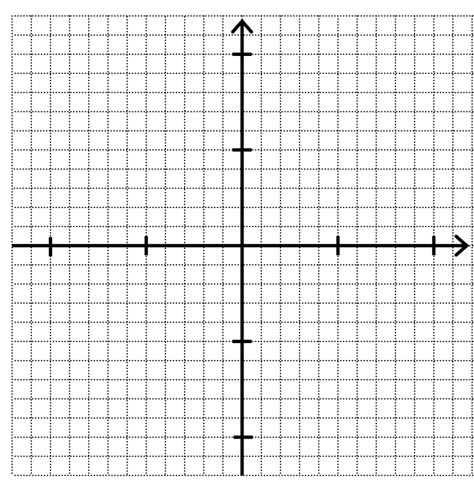
b) Find the common solution to the two equations by using algebra.

c) Graph each equation (on the same graph).

d) What is the common point on the graph?

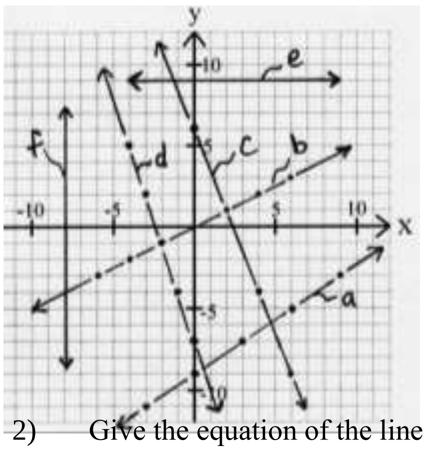






Problem Set #4

1) Give the equation of each line both in slope-intercept form and in standard form.



2) Give the equation of the line that…

a) Has a slope of –3 and passes through (–2,7).
b) Has a slope of ⅔ and a y-intercept of 5.
c) Passes through (–5, –3) with a y-intercept of 2.
d) Passes through the points (6,5) and (3,4).
e) Passes through the points (6,5) and (–3,–7).
f) Passes through the points (6,5) and (2,2).
g) Passes through the points (6,5) and (–4,–3).

3) Give three other equations that have the same solutions as $y = \frac{1}{2}x + 3$

Three Methods

Here are the three common methods for finding the common solution to two linear equations:

- The *substitution method*
- The *graphing method* (as done at the end of the previous problem set.)
- The *linear combination method.* This method may be new to you, so here is an example:

Example: Use the *linear combination method* to find the common solution to these two equations:

$2x + 3y = 4$
$3x – 4y = 23$

Solution: We can choose to either have the x's cancel or the y's cancel. In this case, we will choose to cancel the x's. To do this, I multiply the top equation by 3, and the bottom by –2. So now the equations are:

$6x + 9y = 12$
$-6x + 8y = -46$

It is important to realize that these two equations are equivalent (i.e., they have the same solutions) as the original two equations.

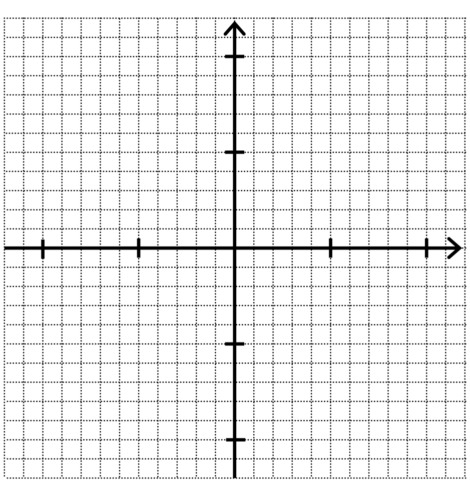
Here's the key: we simply add the two equations together, and the x's cancel. That's why we changed the equations to begin with!

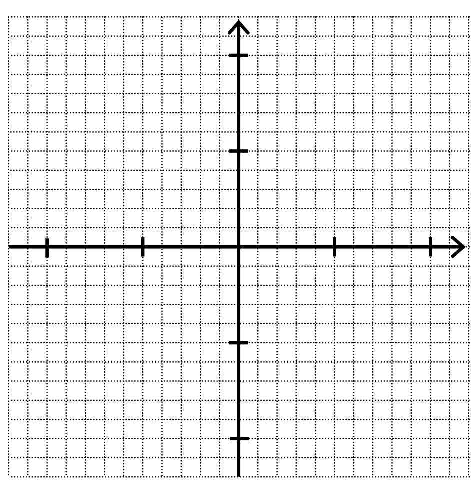
Now we have: $17y = -34$

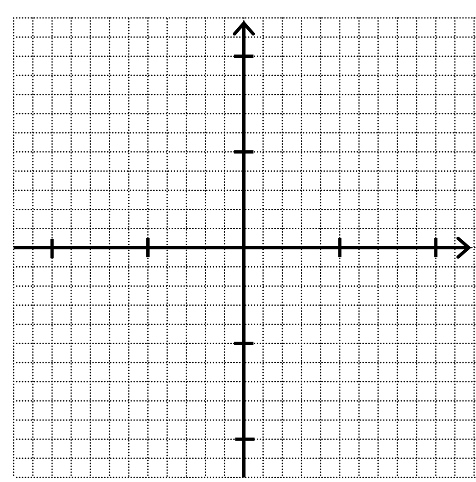
This gives us $y = -2$ as a solution, and by substituting in for y, we get $x = 5$. The common solution to the two original equations is (5, –2).

4) Use each of the three methods to find the common solution to

$2y + 3x = -10$
$6y – 5x = 26$

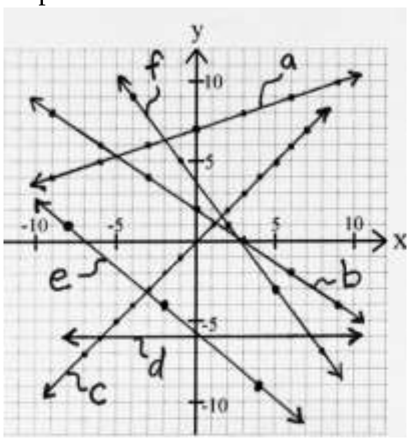






Problem Set #5

1) Give the equation of each line both in slope-intercept form and in standard form.



2) Graph each equation.

a) $y = \frac{3}{4}x + 2$
b) $y = -2x + 7$
c) $y + 2x = 7$
d) $x + 2y = 7$
e) $3x - 5y = 10$
f) $3y + 2x = 5$
g) $y = -x$
h) $x = 4$
i) $y = -x^2 + 6x - 9$

3) Consider the equation $2x - 3y = 12$.

a) What is the slope of its graph?
b) What is the y-intercept?
c) What is the x-intercept?
d) Where is $x = -3$?
e) Give three solutions to the equation.
f) For which point is the value of x and y the same?
g) What solution does it have in common with $x + 3y = 15$

Temperature Conversions

4) The formula for converting from Celsius to Fahrenheit is:

$$F = \frac{9}{5} \cdot C + 32$$

a) What does the 32 indicate?
b) What does the $^9/_5$ indicate?

Use a full-size sheet of graph paper to graph the above equation. The vertical axis should be F, and the horizontal axis should be C. Both axes should have a range from –100 to 100.

Use this graph to estimate the answers to the following:

c) Convert 95°F to °C
d) Convert 10°C to °F
e) Convert 43°F to °C
f) Convert 43°C to °F

5) The formula for converting from Fahrenheit to Celsius is:

$$C = \frac{5}{9} \cdot (F - 32)$$

Multipying in gives us:

$$C = \frac{5}{9}F - 17\frac{7}{9}$$

a) What does the $17\frac{7}{9}$ indicate?
b) What does the $^5/_9$ indicate?

Use a full-size sheet of graph paper to graph the above equation. The vertical axis should be C, and the horizontal axis should be F. Both axes should have a range from –100 to 100.

Use this graph to estimate the answers to the following:

c) Convert 95°F to °C
d) Convert 10°C to °F
e) Convert 43°F to °C
f) Convert 43°C to °F

6) a) If the graphs from problems #4 and #5 are super-imposed upon each other, where would the lines meet?

b) What is the significance of this meeting point?

7) Give the equation of the line that…

a) Has a slope of ⅔ and a y-intercept of (0,6).
b) Has a slope of –5 and passes through the point (2,–7).
c) Passes through the points (3,–2) and (–6,–5).
d) Passes through the points (10,6) and (5,4).
e) Passes through the points (10,6) and (4,2).
f) Passes through the point (3,–2) and runs parallel to the line $y = -2x + 9$
g) Passes through the point (–1,4) and is perpendicular to the line $y = \frac{3}{4}x - 5$

8) Use the linear combination method to find the common solution to:

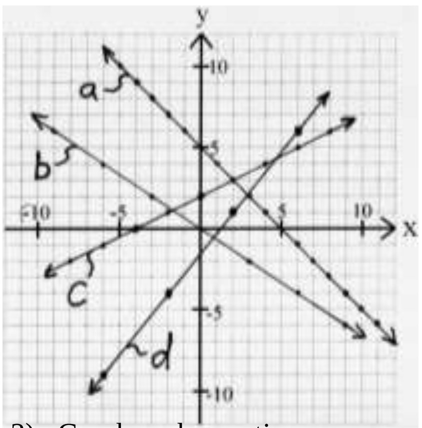
$$5x - 6y = 31$$
$$3x + 4y = -8$$

9) Use each of the three methods to find the common solution to

$$x + 2y = 2$$
$$4x - 3y = 30$$

Problem Set #6

1) Give the equation of each line.



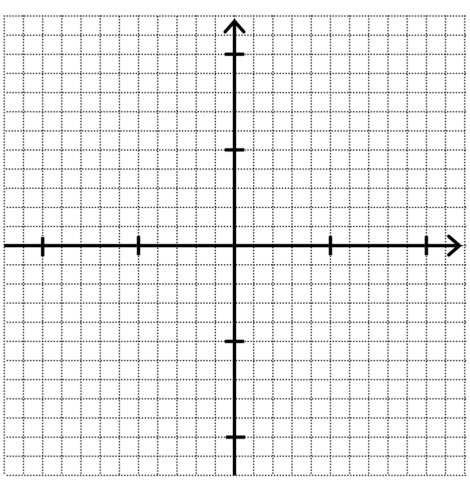
2) Graph each equation.
   a) $y = -\frac{1}{2}x$
   b) $y = \frac{3}{4}x - 5$
   c) $6y + 5x = 18$
   d) $3y + 7x = -18$

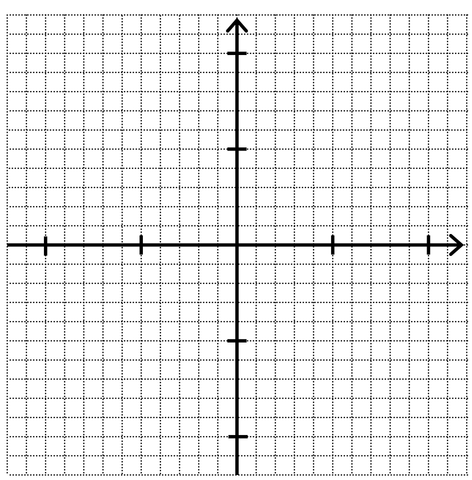
3) Use both the graphing method and the linear combination method to find the common solution to the equations given in #2c and #2d above.

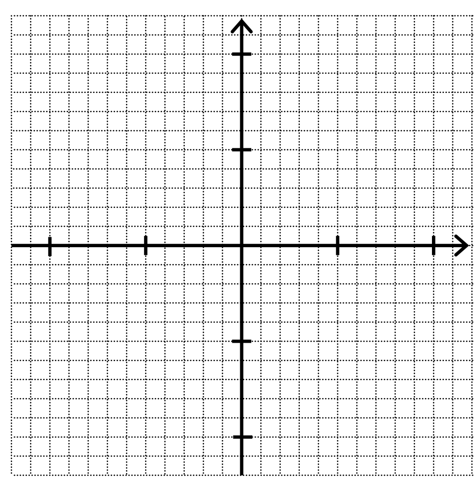
4) Use both the graphing method and the substitution method to find the common solution to the equations given in #2b and #2c above.

5) Give the equation of the line that…
   a) Has a slope of ⅔ and passes through the point (–6,1).
   b) Passes through the points (2,–5) and (6,–7).
   c) Passes through the points (1,7) and (–3,5).
   d) Passes through the point (10,4) and runs parallel to
   $$y = -\frac{2}{5}x + 20.$$
   e) Passes through the point (10,4) and is perpendicular to $y = -\frac{2}{5}x + 20$.

6) Jason currently has $3,400 of debt from an interest free loan, and he has a total of $400 in savings. He decides that starting today he will pay $100 per month toward his debt and that he will also save an additional $150 per month by putting it under his mattress (therefore no interest).
   a) Give an equation that expresses the balance of his debt over time.
   b) Give an equation that expresses his total savings over time.
   c) Graph the two above equations on the same graph.
   d) How much savings will he have after 2 years?
   e) When will he have $1600 in savings?
   f) When will his debt finally be zero?
   g) Where do the two graphs meet? What is the significance of that point?



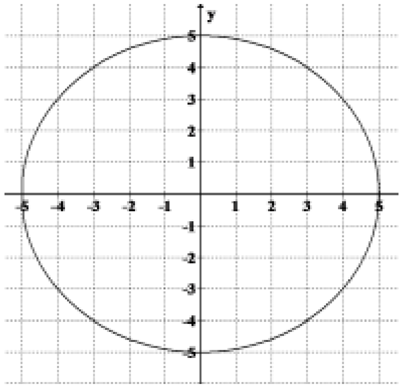




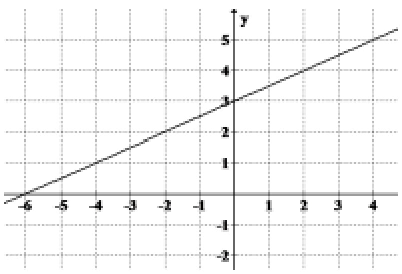
Cartesian Geometry – Part I ANSWERS

Problem Set #1

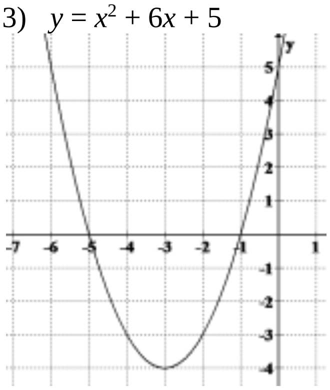
1) $x^2 + y^2 = 25$



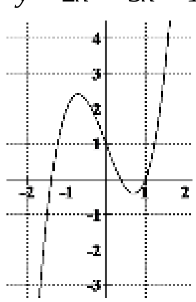
2) $y = \frac{1}{2}x + 3$



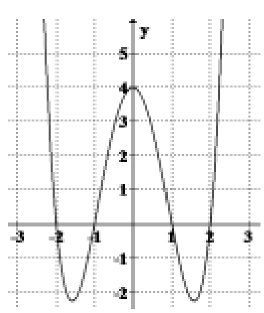
3) $y = x^2 + 6x + 5$



4) $y = 2x^3 - 3x + 1$

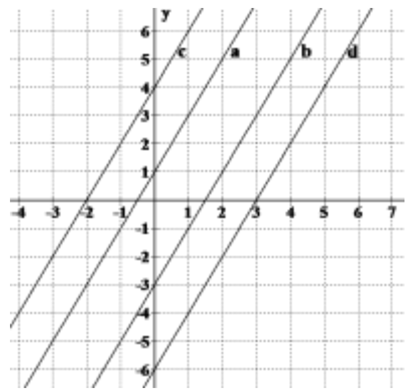


5) $y = x^4 - 5x^2 + 4$



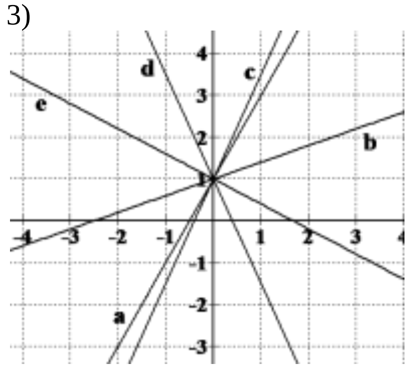
Problem Set #2

1)



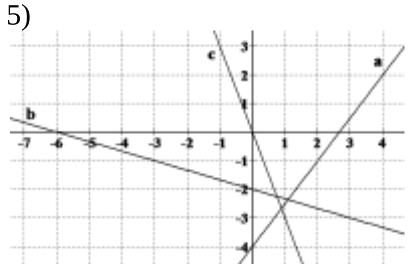
2) Where the line hits the y-axis.

3)



4) How the line tilts (the slope).

5)



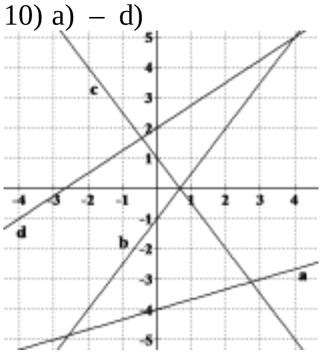
Cartesian Geometry – Part I ANSWERS

Problem Set #3

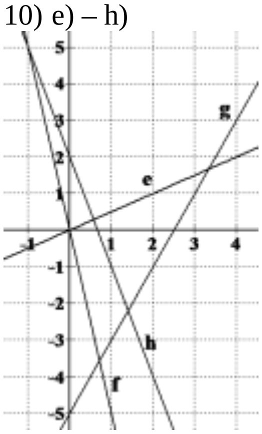
1)

a) 4 b) $-\frac{5}{2}$ c) $\frac{2}{3}$ d) $\frac{5}{2}$ e) $\frac{2}{5}$ f) $\frac{2}{5}$

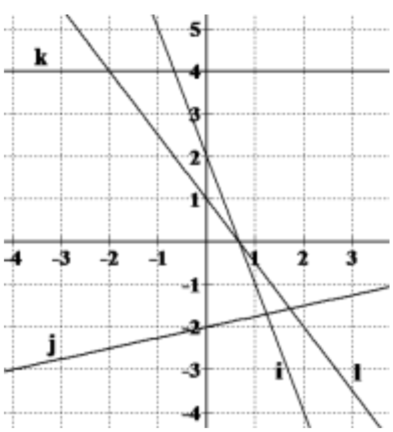
2) Lines with negative slopes point up and to the left (and down and to the right) whereas lines with positive slopes point up and to the right (and down and to the left).
3) 1
4) Its slope is less than 1.
5) Its slope is greater than 1.
6) The slopes are equal.
7) The slopes are opposite reciprocals.
8) 0
9) Undefined or infinite slope.
10) a) – d)



10) e) – h)



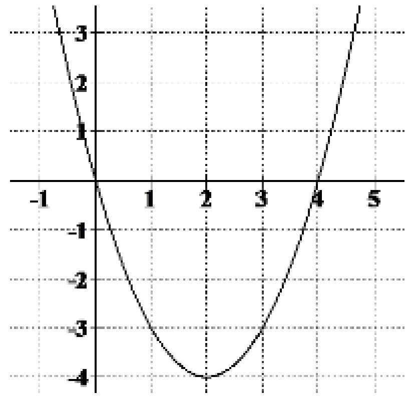
10) i) – l)



11) 10c

12)

a) (0,0), (-1,5), (1, -3) for example.
b) x can be any real number.
c) $y \geq -4$
d)



e) Yes. It shows that the graph spans all of the x values and does not go below -4 on the y-axis.

13)

a) First equation:

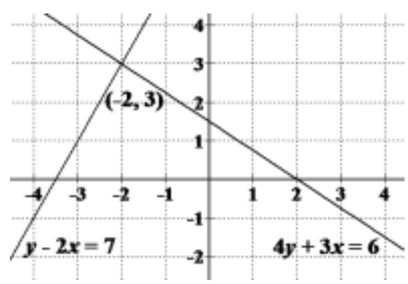
$(0, \frac{3}{2})$, (2, 0), (6, -3)

Second equation:

(0, 7), (1, 9), (-2, 3)

b) (-2, 3)

c)



d) (-2, 3)

Cartesian Geometry – Part I ANSWERS

Problem Set #4

1)

a) $y = \frac{2}{3}x - 9$ or

$-2x + 3y = -27$

b) $y = \frac{1}{2}x$ or $-x + 2y = 0$

c) $y = -\frac{5}{2}x + 6$ or

$5x + 2y = 12$

d) $y = -3x - 7$ or $3x + y = -7$
e) $y = 9$
f) $x = -8$

2)

a) $y = -3x + 1$

b) $y = \frac{2}{3}x + 5$

c) $y = x + 2$

d) $y = \frac{1}{3}x + 3$

e) $y = \frac{4}{3}x - 3$

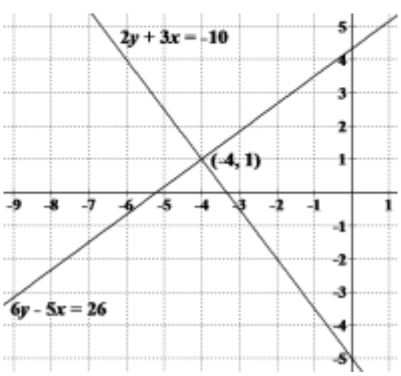
f) $y = \frac{3}{4}x + \frac{1}{2}$

g) $y = \frac{4}{5}x + \frac{1}{5}$

3) $-\frac{1}{2}x + y = 3$ or $-x + 2y = 6$ or

$2y - x - 6 = 0$ for example.

4)



Problem Set #5

1)

a) $y = \frac{1}{3}x + 7$ or $-x + 3y = 21$

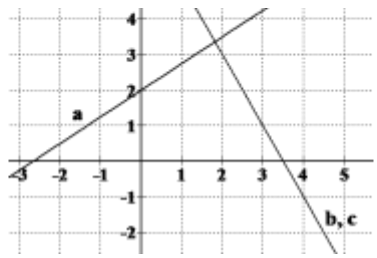
b) $y = -\frac{2}{3}x + 2$ or $2x + 3y = 6$

c) $y = x$
d) $y = -6$

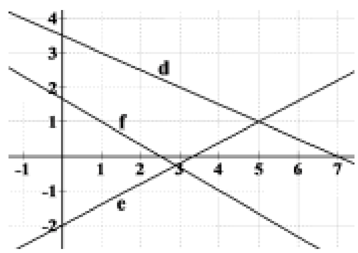
e) $y = -\frac{5}{6}x - \frac{17}{3}$ or $5x + 6y = -34$

f) $y = -\frac{4}{3}x + \frac{11}{3}$ or $4x + 3y = 11$

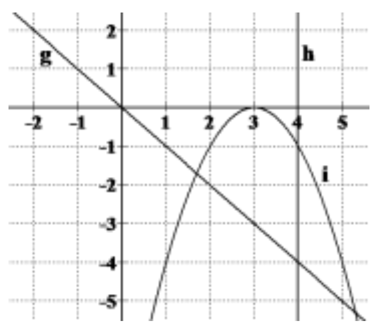
2) a) – c)



2) d) – f)



2) g) – i)



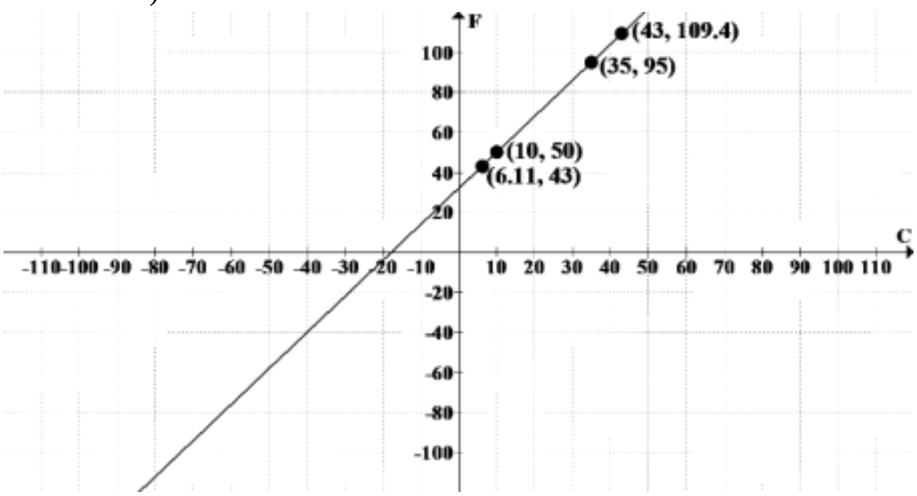
3)

a) $\frac{2}{3}$

b) -4 or (0, -4)
c) 6 or (6, 0)
d) $y = -6$ where $x = -3$
e) (0, -4), (6, 0), (-3, -6) for example.
f) (-12, -12)
g) (9, 2)

Cartesian Geometry – Part I ANSWERS

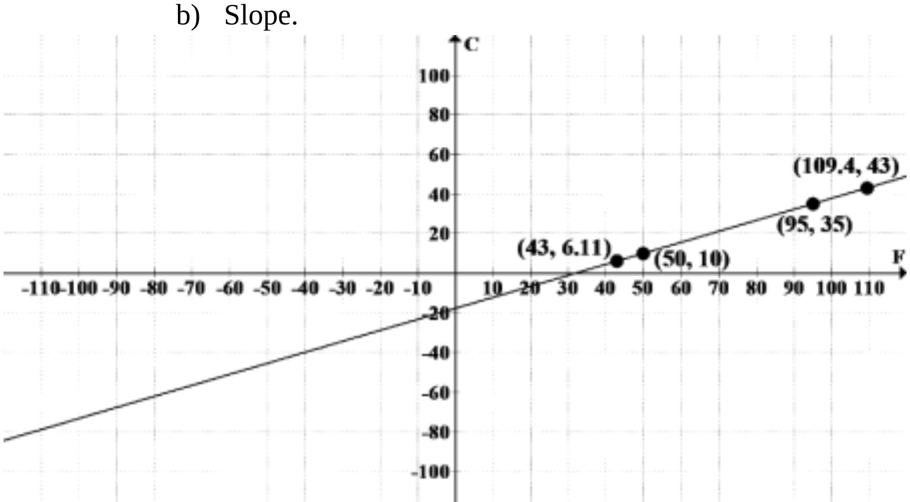
4)



a) y-intercept. This shows the value of F when C = 0.
b) Slope. An increase in 5°C is the same as an increase of 9°F.
c) 35° C
d) 50° F
e) $6\frac{1}{9}$°
f) $109\frac{2}{5}$°

5)

a) y-intercept which is $\frac{5}{9} \cdot 32$.
b) Slope.



c) 35° C
d) 50° F
e) $6\frac{1}{9}$°
f) $109\frac{2}{5}$°

6)

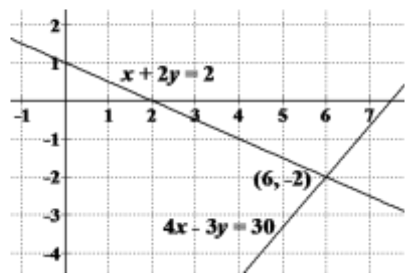
a) -40°
b) -40°C is equal to -40°F.

7)

a) $y = \frac{2}{3}x + 6$
b) $Y = -5x + 3$
c) $y = \frac{1}{3}x - 3$
d) $y = \frac{2}{5}x + 2$
e) $y = \frac{2}{3}x - \frac{2}{3}$
f) $y = -2x + 4$
g) $y = -\frac{4}{3}x + \frac{8}{3}$

8) $(2, -\frac{7}{2})$

9) (6, -2)



Cartesian Geometry – Part I ANSWERS

Problem Set #6

1)

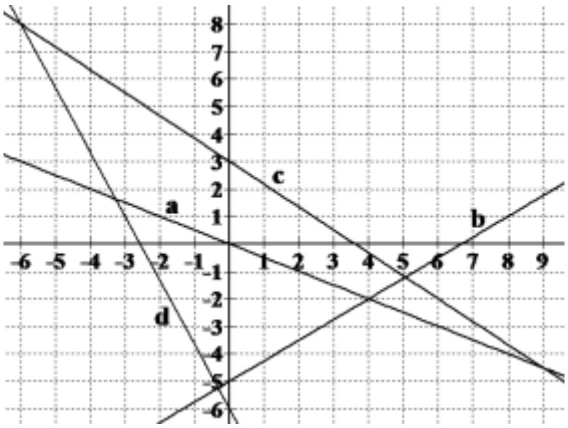
a) $y = -x + 5$

b) $y = -\frac{2}{3}x$

c) $y = \frac{1}{2}x + 2$

d) $y = \frac{5}{4}x - \frac{3}{2}$

2)



3) (-6, 8) See graph inproblem 2.

4) $(\frac{96}{19}, -\frac{23}{19})$ or $(5\frac{1}{19}, -1\frac{4}{19})$

See graph in problem 2.

5)

a) $y = \frac{2}{3}x + 5$

b) $y = -\frac{1}{2}x - 4$

c) $y = \frac{1}{2}x + \frac{13}{2}$

d) $y = -\frac{2}{5}x + 8$

e) $y = \frac{5}{2}x - 21$

6) a) $y = -100x + 3400$

b) $y = 150x + 400$

c) See bottom of page

d) $4000

e) 8 months

f) After 34 months.

g) At (12, 2200). Jason’s net worth is $0 at this point.