A (very abbreviated) summary of The Great Story

I tell my students that the story of the "foundational crisis" is one of the greatest stories ever told. It has complex characters, conflict, struggle, disappointment, and hope. The plot is dramatic, and the conclusion is so unexpected and shocking that, even today, much of the mathematical community seems to operate under the premise that it never really happened.

ACT I – Perfect truth (300 BC – 1820 AD)

We already set the stage for the story in tenth grade – going back some 2300 years – when we studied Euclid's *Elements*. Euclid's work is an axiomatic system comprised of 465 theorems, and encompasses much of the known mathematics at that time. The truth of these theorems rests upon five postulates, or self-evident truths. Because of the truth of these five postulates, and the flawless logic that is followed with each proof, *The Elements* is considered to be the perfect model of truth. It is *THE* math textbook for over 2000 years. To a large degree, it is the foundation upon which the rest of the mathematical world is built. This is where we leave things with the tenth grade, and that was where the world of mathematics found itself at the start of the 19th century – very comfortably believing that Euclid was unshakeable, and therefore mathematical truth was irrefutable.

ACT II – The fall of Euclid (1820 – 1870)

But what about the fifth postulate? It certainly isn't as obvious as the first four postulates. The famous fifth postulate (known as the *Parallel Postulate*) reads:

> *If a line falling on two lines makes the interior angles on the same side less than two right angles, the two lines, if produced indefinitely, meet on that side on which the angles are less than two right angles.*

Certainly, if you take the time to make a drawing and think it through, then it is clearly true. But wouldn't *The Elements* be even better – wouldn't the foundation be even stronger – if we could prove the fifth postulate based upon the first four postulates? For centuries, mathematicians try to prove the fifth postulate. But nobody succeeds. And then, the unthinkable happens. Not suddenly, but slowly. Slowly over the first few decades of the 19th century, the mathematical world comes to the cruel realization that the fifth postulate *is not necessarily true*. We can assume it to be true, and then we end up with Euclid. But we can also assume something completely different and contradictory to the fifth postulate, and then we end up with a completely different *and equally valid* geometry. This opens the door for the development of new geometries (e.g., projective geometry and non-Euclidean geometry).

ACT III – New hopes are shattered (1870 – 1902)

As we enter the second half of the 19th century, Euclid is no longer the only possible truth; the unshakable foundation is gone. In place of the ancient Greeks' reliance on geometry, much of mathematics now relies on algebra, or more properly: formal analysis (calculus, differential equations, and functional analysis). But what is the foundation for analysis? Enter stage left: the rise of formal logic. There is hope that the foundation can be rebuilt. Georg Cantor's *set theory* provides some of the tools needed for the new field of formal logic. Cantor's ideas are quite controversial – are there really different "sizes" of infinity?

But it's Gottlob Frege who takes the torch and runs with it. Frege wants to show that mathematics grows out of logic. His world of logic is destined to become the new foundation for mathematics. In the process, Frege invents axiomatic predicate logic. His monumental works (*Begriffsscrift* and *Grundgesetze der Arithmetik*) take more than 25 years to complete. Certainly, his time would be well spent if it indeed became the new, unshakeable foundation. But just as Frege is laying the last few bricks (his final book was about to go to press), Bertrand Russell writes his famous letter to Frege (in 1902), informing him of a paradox within his work. Frege is understandably crushed as is shown by the comment he adds to the appendix of the book: "Hardly anything more unfortunate can befall a scientific writer than to have one of the foundations of his edifice shaken after the work is finished."

ACT IV – Stormy times and more uncertainty (1902 – 1922)

It seems that a powder keg has been ignited. David Hilbert, one of the foremost mathematicians of the time, sums it up nicely, stating: "The present state of affairs where we run up against the paradoxes is intolerable. If mathematical thinking is defective, where are we to find truth and certitude?" Mathematicians even begin to doubt if there is any truth behind mathematics. Bertrand Russell expresses a dramatic sentiment shared by several others: "Mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true." (He is referring to axioms that are no longer self-evident truths, and to "primitive undefined terms".)

Has mathematics really become just a meaningless game?

The mathematical world becomes bitterly divided into philosophical camps. The *Logicists* are led by Bertrand Russell, who is determined to pick up the pieces of Frege's work and see it to a successful completion. David Hilbert, the powerful leader of the *Formalists*, believes that the real solution is through a formal proof to show the consistency of arithmetic. And then there are the *Intuitionists* – the hecklers and thorns-in-the-sides of the formalists and logicists. Many of the intuitionists are great and well-respected mathematicians, such as Leopold Kronecker, Henri Poincaré, and L.E.J. Brouwer. But the intuitionists stand philosophically in direct contrast to the likes of Russell and Hilbert. Kronecker believes that whole numbers were given to us by God. Poincaré says: "Arithmetic cannot be justified by an axiomatic foundation. Our intuition precedes such a structure." And Brouwer insists: "Our intuition, not logic or experience, determines the soundness and acceptability of ideas."

Hilbert and Russell feel a sense of deep responsibility to save mathematics from the heresy of the intuitionists. Hilbert's "program" aims to prove that arithmetic (the theory of natural numbers) is *consistent*. Russell and Whitehead, on the other hand, work tirelessly for ten years on their version of what Frege had started. Their three-volume work, *Principia Mathematica* (published in 1910-13), is written in the language of formal logic. Nobody can wade through the 2000-page tome, let alone judge whether it has successfully achieved its goal. Is *Principia Mathematica* both *consistent* and *complete*?[1] Has it provided a new, unshakable foundation for mathematics? For nearly 20 years, nobody really knows the answer to that question.

ACT V – An Unexpected Ending (1922 – 1931)

As we enter the 1920's, the uncertainty and precarious position of mathematics has not improved. The tension between Hilbert and Brouwer grows more tense. In 1922, a group of mathematicians and philosophers begins to meet weekly in a Vienna café. The "Vienna Circle" soon gains influence over many intellectuals of the time. Their new brand of philosophy (later called *Logical Positivism*) pronounces as "meaningless" any statement that cannot be supported with empirical evidence. Mathematics, in their eyes, is not about truth at all; it fact it is completely devoid of meaning whatsoever. The only real purpose of math is as a language to be used in science. The Vienna Circle has an unlikely guru: the young, radical philosopher, Ludwig Wittgenstein. He publishes a work, *Tractatus Logico Philosophicus*, which is intended to "resolve for good all problems of philosophy".

In the late 1920's, an introverted graduate student and logician, Kurt Gödel, attends the Vienna Circle's weekly meetings. He sits quietly on the side, takes it all in, and slowly becomes more and more convinced that the Vienna Circle's philosophy is desperately wrong. Gödel is a Platonist; he believes that math is meaningful. And he wants to prove it! He stops attending the Vienna Circle meetings in 1928, and works on his proof for three years. His (now famous) *Incompleteness Theorems* state:

- Any formal, axiomatic system that encompasses arithmetic (such as *Principia Mathematica*) is essentially incomplete.
- If *Principia Mathematica* is consistent, then the consistency cannot be proven within that system.

Principia Mathematica suffers a fatal blow. Gödel's theorems dramatically and unexpectedly put an end to both the foundational crisis and the 2,400-year-long "epic search for truth". Hilbert and Russell wanted to give a firm foundation for all mathematics by eliminating intuition from mathematics. Gödel's proof seems to show that math cannot proceed without intuition. He believes that mathematical reality must exceed all formal attempts to contain it.

There is no unshakable foundation for mathematics to replace Euclid's *Elements*. What are the implications?

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[1] an axiomatic system is said to be "consistent" if it is free of any contradictions. It is "complete" as long as every theorem (or statement or formula) that is true can be proven to be true within that system.

Epilogue (1931 – present)

Many people misunderstood the implications of Gödel's proof. Gödel "defeated" Russell and Hilbert, but sadly, the logical positivists (Vienna Circle) saw it as further evidence of the meaninglessness of math, and this attitude continues to have an impact on mathematics education today.

After Sputnik (the Soviets' first satellite in 1957), America became focused on beating the Soviets. Through this narrow lens, the purpose of mathematics education became to produce more scientists in an effort to win this mad race. Over the past few decades, the world has changed in many ways – socially, economically, and politically. Perhaps now more than ever, there is a realization that mathematics education needs to be overhauled. Opinions differ about where to place the blame, and how to fix it. New mathematics curricula have come and gone. Math textbooks are thicker, "prettier" (more graphics and fancier "packaging"), and more expensive – but less effective than ever. The trends are clear: an increase in teaching to standardized tests; more reliance on technology-driven learning; and teaching more advanced material at a younger and younger age.

And the results? A superficial understanding of material; a loss of a "sense of number"; an inability to do simple calculations in your head; no time left for depth, discovery, or true problem solving. In short, mathematics has become meaningless for most students.

But we must not give up. We all need to become math missionaries. We need to embark on a crusade to make math meaningful.