# Gödel's Proof – Gödel Numbers

## Table of (some of the) Symbols within PM and *Gödel numbering*

| Symbol | Gödel # | Meaning | Symbol | Gödel # | Meaning |
|---|---|---|---|---|---|
| ~ | 1 | not | ( | 8 | left parenthesis |
| ∨ | 2 | or | ) | 9 | right parenthesis |
| ⊃ | 3 | if…then… | , | 10 | comma |
| ∃ | 4 | there exists… | + | 11 | plus |
| = | 5 | equals | · | 12 | multiplication |
| 0 | 6 | zero | x | 13 | variable |
| s | 7 | successor of | y | 17 | variable |
| | | | z | 19 | variable |

**Note:** ss0 stands for 2.

## Vocabulary

*PM*: the given formal axiomatic system within which all the axioms, definitions, and rules are given. PM is assumed to be the system given in *Principia Mathematica* (by Russell and Whitehead), but could be any formal axiomatic system that includes the basic properties of arithmetic (e.g., addition, multiplication).

*formula*: a well-formed formula (wff) or statement within the system PM that is made according to the rules. A formula may be either true or false. A formula becomes a *theorem* once it is proven. "x =" is not a proper formula. Examples of proper formulas include:

x = ssss0, which means "x equals 4"
sssss0 = ss0 + ss0, which means "5 equals 2 + 2"
(∃x)(x = ss0 + sss0), which means "there exists an x such that x equals 2+3"
(∃x)(x = ss0) ⊃ (∃y)(~(y = 0)), which means "If there exists x such that x equals 2, then there exists y such that y is not equal to 0."

*proof*: A sequence of formulas, each one of which is justified according to the rules and axioms, and the last formula in the proof is the theorem that is to be proven.

## Gödel numbering:

- The key idea is that any possible formula or proof can be coded as a unique Gödel number, and any Gödel number can be decoded to get the formula or proof that it represents.
- To determine what a Gödel number represents, we break it down into its prime factorization.
- The bases of the prime factorization of any Gödel number are always consecutive prime numbers.
- The Gödel number of a single formula is such that each exponent in the prime factorization tells us a symbol in the formula.
- The Gödel number of a proof – or perhaps an entire system of proofs – is such that each exponent in the prime factorization is itself the Gödel number of a single formula.

Question: What is the formula represented by the Gödel number 31104000000?
Solution: We break down the number into its prime factorization, which is $2^{13}\cdot3^{5}\cdot5^{6}$. The exponents tell us that the three symbols are "x", "=", and "0". The formula is "x = 0".

Question: What is the Gödel number for the formula "z = ss0 + y" ?
Solution: The Gödel number is given by multiplying out $2^{19}\cdot3^{5}\cdot5^{7}\cdot7^{7}\cdot11^{6}\cdot13^{11}\cdot17^{17}$.

Question: What is the Gödel number for the formula (∃x)(x = ss0) ⊃ (∃y)(~(y = 0)) ?
(The solution is left for you to figure out!)

Question: What is the Gödel number for this proof?: x = sss0
y = x
y = sss0

Solution: The first formula has the Gödel number $2^{13}\cdot3^{5}\cdot5^{7}\cdot7^{7}\cdot11^{7}\cdot13^{6}$, which we will call a.
The second formula has the Gödel number $2^{17}\cdot3^{5}\cdot5^{13}$, which we will call b.
The third formula has the Gödel number $2^{17}\cdot3^{5}\cdot5^{7}\cdot7^{7}\cdot11^{7}\cdot13^{6}$, which we will call c.
Therefore, the whole proof has the Gödel number $2^{a}\cdot3^{b}\cdot5^{c}$

Question: What is the Gödel number for the whole of *Principia Mathematica*?
(This is left for you to imagine!)

# Gödel's Proof – Functions & Formulas

## I. Key Ideas

1. *Truth and Certainty*. Whether a statement is true or not is a different matter than whether or not the statement can be proven as true or false. For example, the statement "Lions are mammals" may be true, but it cannot be proven within PM.
2. *Meta-Mathematics*. Meta-mathematics is when we "stand above" a mathematical system and make statements about it. Examples of meta-mathematical statements include: "Euclid proved 465 theorems in *The Elements*"; "PM is consistent." Quite amazingly, Gödel found a way to create formulas within PM that were, at the same time, meta-mathematical statements about PM.
3. *If a formula and its negation (opposite) can both be proven as true, the system is inconsistent.*
4. *If a system is inconsistent, then any formula (even one that is obviously false) can be proven as true*. For example, *if* it could be proven within a system that $2+2=4$ and that $2+2\neq 4$, *then* there is a contradiction, and therefore the system is inconsistent. It would then follow that any formula within the system could be proven as true (e.g., $4=5$; $7=8$; $21=24$, etc.).
5. *If, within a given mathematical system, there exists a (presumably false) formula that cannot be proven as true, then the system is consistent.* This is the contrapositive of the above statement.
6. *In order to prove that a system is consistent, we only need to find one (presumably false) formula within the system that cannot possibly be proven as true.* For example, if we can prove, by using the axioms of the system, that there is no possible way to prove that $3+9=7$, then we have proven that the system is consistent. (Note that because the consistency of PM is under question, simply proving that $3+9 = 7$ is false is not sufficient to be able to conclude that it can't be proven true.)
7. *Given the statement: "If E is true then F must be true", if we happen to know that F is false, then we can conclude that E cannot be true.*
8. *With any mathematical system, if there exists one true formula that is not provable within the system, then that system is incomplete.*

## II. Functions. 
Gödel found a way to express both of the following functions within PM.

Dem(z,w) "Dem" is short for "demonstrate", which is another word for "prove". It answers the question: "Is z a proof of w?" This function is true only if the sequence of formulas given by the Gödel number z is a proof of the formula given by the Gödel number w. (Formula w would be the last formula in proof z.)

Sub(a,b,c) "Sub" is short for "substitute". This is a bizarre but important function. a is the Gödel number of a formula. b is the Gödel number of a symbol. c simply represents a symbol. Sub(a,b,c) tells us to take the formula with Gödel number a, find all occurrences of the symbol indicated by Gödel number b, and then replace them with the symbol c. The function Sub(a,b,c) returns the Gödel number of the resulting formula.

**Example:** If the formula $x = y + ss0$ is given by the Gödel number k, then sub(k,13,0) returns the Gödel number of the formula $0 = y + ss0$.

## III. Formulas. 
The key to Gödel's brilliance is that he finds a way to express certain important meta-mathematical statements about PM within the system of PM. He does this by using two amazing statements – each one is a meta-mathematical statement that is expressed within PM. (Note that for now, we are not saying whether either formula is true, or not.)

Formula B: (∃w) ~(∃z) Dem(z,w)

This reads as: "There exists a formula w such that there does *not* exist a proof (z) of w", or: "w is not provable". Using Key Idea #5, above, Formula B is therefore the equivalent of the meta-mathematical statement: **"PM is consistent."**

Formula G: ~(∃x) Dem(x,[sub(n,17,n)])

This statement is very important, but difficult to comprehend. You must first know that n is the Gödel number of the formula ~(∃x) Dem(x,[sub(y,17,y)]). Formula G then reads, "There does *not* exist an x such that x is a proof of the formula given by the Gödel number that results from sub(n,17,n)." In other words, the statement that results from sub(n,17,n), *which is Formula G itself*, is not provable. Formula G is therefore the equivalent of the meta-mathematical statement: **"Formula G is not provable."**
(Note that at this point we don't know whether G is true or not.)

The two above formulas are used as centerpieces in Gödel's proof.

# Gödel's proof – The Central Argument

Note: This proof uses formulas B and G (see previous sheet, *Gödel's Proof – Functions & Formulas*). What follows is only an outline of the central argument of the proof.[1]

The following steps are justified by using formulas, axioms, and rules from within PM.

1. Assumption: PM is consistent, which means Formula B is true.
2. G says, "G is not provable." Its negation, ~G, says, "G is provable".
   - (For a moment, ignoring what G says) if, somehow, G can be proven, then G is provable, which is what ~G says, so it has also been proved. In summary, *if G is provable, then ~G is provable*.
   - If, somehow, ~G can be proven, then because ~G says "G is provable", G must also be provable. In summary, *if ~G is provable, then G is provable*.

   (Gödel proved the two above statements within the system of PM.)
3. We just said that *if* G is provable, then ~G is provable. We also know (from Key Idea #3, previous page) that "*If a formula and its negation can both be proven as true, the system is inconsistent.*" Then it follows that if G is provable, then the system (PM) is inconsistent. This allows us to conclude (using Key Idea #7, previous page): If PM is consistent, then *G is not provable*.

Now, using meta-mathematical reasoning…

4. We have just shown that G is not provable (assuming that PM is consistent). But, G says "G is not provable." So now we know that *G is true*. Therefore, *G is true, but not provable within PM*.
5. We have now found a statement, expressible within PM, that is true but not provable. We also know that: "*With any mathematical system, if there exists one true formula that is not provable within the system, then that system is incomplete.*" (Key Idea #3, previous page)

**Conclusion #1:** *If PM is consistent, then it is incomplete*.[2, 3]

6. The statement "PM is incomplete" is the equivalent of saying: "There exists one true formula of PM that is not provable." We have now discovered that G is such a formula. Therefore, we can now say that G has the following new meaning: "*PM is incomplete*".
7. Assumption: Formula B (which says: "PM is consistent") is provable.
8. We now can say the following:
   - From Conclusion #1: "If PM is consistent, then PM is incomplete"
   - Which also means: "If B [is true], then G [is true]."
   - Which leads us to: "If B is provable, then G is provable."
9. In summary, if we assume that PM is consistent (B is true, step #1) and assume that the consistency is provable (B is provable, step #7), then it must follow (from step #8) that *G is provable*.
10. However, we said in step #3 that if PM is consistent, then G is *not* provable. We have a contradiction, which means that our assumption (step #7) is incorrect.
    *Therefore B* (which says: "PM is consistent") *is not provable*.

**Conclusion #2:** *If PM is consistent, then the consistency cannot be proven.*[4]

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[1] The teacher should read *Gödel's Proof* (by Ernest Nagel and James Newman, NYU Press, 2001, 2nd edition) for an excellent account of Gödel's Proof.

[2] Gödel also proves that PM is *essentially* incomplete, which means that even if the system is "repaired" by adding more axioms so that it can handle any problematic formulas (like G), then another true, but unprovable formula can always be constructed. And Gödel proves that this is true of *any* formal, axiomatic system that encompasses the elementary properties of whole numbers, including addition and multiplication.

[3] The meta-mathematical statement, "If PM is consistent then PM is incomplete" (step #5), can be expressed within PM by the formula (which can be abbreviated $B \supset G$):

$$[(\exists w) \sim(\exists z)\, \mathrm{Dem}(z,w)] \supset [\sim(\exists x)\, \mathrm{Dem}(x,[\mathrm{sub}(n,17,n)])]$$

[4] To be more precise, I quote Ernst and Nagel (p107): "If PM is consistent, its consistency cannot be proven by any meta-mathematical reasoning that can be mirrored within PM itself. It does *not* exclude [the possibility of] a meta-mathematical proof of the consistency of PM. What it excludes is a proof of consistency that can be mirrored inside of PM…Proofs that cannot be mirrored inside the systems that they concern are not finitistic; they [therefore] do not achieve the proclaimed objectives of Hilbert's original program."