Proofs of the Summation Formulas

The formulas are (for i = 1 to n):

$\Sigma i = \frac{n(n+1)}{2}$; $\Sigma i^2 = \frac{n(n+1)(2n+1)}{6}$; $\Sigma i^3 = \frac{n^2(n+1)^2}{4} = (\Sigma i)^2$; $\Sigma i^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

Here are two ways that these formulas can be proven:

1. Mathematical Induction

For example, proving Σi^2:

Step 1: We must show that the formula is valid for n = 1. If n=1 then $\Sigma i^2 = 1^2 = 1$.

The formula is $\frac{n(n+1)(2n+1)}{6}$, which for n=1 gives a result of 1. Therefore it is valid for n=1.

Step 2: We must now show that, assuming that the formula is valid for some integer k, that it is also valid for k+1.

Putting k+1 into the formula, we get $\frac{(k+1)(k+2)(2k+3)}{6}$.

To show this is equal to the sum of the squares of all the numbers from 1 to k+1, we get:

$$(1^2 + 2^2 + 3^2 + 4^2 + ... + k^2) + (k+1)^2$$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

factoring out (k+1) as GCF, we get:

$$= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6}$$

$$= \frac{(k+1)[2k^2 + 7k + 6]}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$ therefore it works for k+1

$\therefore$ We have now proven that the formula is correct for all integer values of $k \geq 1$.

2. Proof by summing equations

For example, in proving Σi^4, we use the identity $x^5 - (x-1)^5 = 5x^4 - 10x^3 + 10x^2 - 5x + 1$.

We create n equations by first plugging 1 into x in the above identity, then we create a second equation by plugging in 2 for x, etc., all the way up to plugging in n for x. All of this gives us the following n equations:

$$1^5 - 0^5 = 5\cdot1^4 - 10\cdot1^3 + 10\cdot1^2 - 5\cdot1 + 1$$
$$2^5 - 1^5 = 5\cdot2^4 - 10\cdot2^3 + 10\cdot2^2 - 5\cdot2 + 1$$
$$3^5 - 2^5 = 5\cdot3^4 - 10\cdot3^3 + 10\cdot3^2 - 5\cdot3 + 1$$

etc.

$$(n-1)^5 - (n-2)^5 = 5(n-1)^4 - 10(n-1)^3 + 10(n-1)^2 - 5(n-1) + 1$$
$$n^5 - (n-1)^5 = 5n^4 - 10n^3 + 10n^2 - 5n + 1$$

Now add up all n equations (notice that the left sides mostly cancels). We get:

$$n^5 = 5(1^4+2^4+3^4 + ... + n^4) - 10(1^3+2^3+3^3 + ... + n^3) + 10(1^2+2^2+3^2 + ... + n^2) - 5(1+2+3 + ... + n) + n$$

which is: $n^5 = 5(\Sigma i^4) - 10(\Sigma i^3) + 10(\Sigma i^2) - 5(\Sigma i) + n$

$$5(\Sigma i^4) = n^5 + 10\frac{n^2(n+1)^2}{4} - 10\frac{n(n+1)(2n+1)}{6} + 5\frac{n(n+1)}{2} - n$$

$$\Sigma i^4 = \frac{6n^5 + 15n^4 + 10n^3 - n}{30}$$

which factors to: $\Sigma i^4 = \frac{n(n+1)(2n+1)(3n^2 + 3n - 1)}{30}$