Calculus – Part I ANSWERS

Problem Set #1

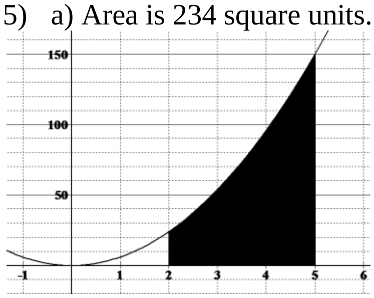
1)
- a) f '(x) = $5x^4$
- b) f '(x) = 6
- c) f '(x) = 0

2)
- a) $\frac{dy}{dx} = -8x^{-3} = -\frac{8}{x^3}$
- b) $\frac{dy}{dx} = -\frac{8}{x^3}$

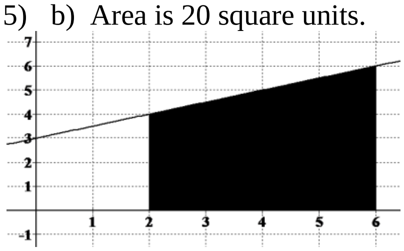
3)
- a) $6x + 8$
- b) $4x^3 - 18x^2 + 4$
- c) $35x^6 + 3x^2 + \frac{3}{4x^4}$

4)
- a) $F(x) = \frac{1}{6}x^6 + C$
- b) $F(x) = 3x^2 + C$
- c) $F(x) = 8x + C$
- d) $F(x) = x^3 + 4x^2 + C$
- e) $F(x) = \frac{1}{5}x^5 - \frac{3}{2}x^4 + 2x^2 - 5x + C$
- f) $F(x) = -\frac{4}{x} + C$

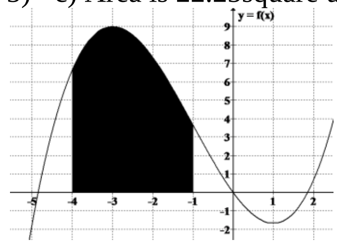
5) a) Area is 234 square units.



5) b) Area is 20 square units.



5) c) Area is 22.25square units



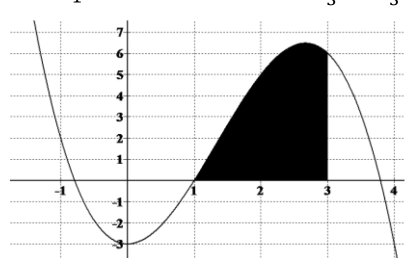
6) a)

x	f(x)	f '(x)
5	-28	-35
4	-3	-16
3	6	-3
2	5	4
1	0	5
0	-3	0
-1	2	-11
-2	21	-28

- b) The function f evaluated at 3 equals 6 or we can loosely say that when x = 3, y = 6.
- c) When x = 3, the slope of f(x) is -3.
- d) 1, $\frac{3 \pm \sqrt{21}}{2}$
- e) (0, -3) & ($\frac{8}{3}$, $\frac{175}{27}$)

6) f) – g)

$$\int_1^3 -x^3 + 4x^2 - 3\, dx = \frac{26}{3} = 8\frac{2}{3}$$



Calculus – Part I ANSWERS

7) b	8) a	9) d	10) c	11) a	12) c
13) c	14) d	15) a	16) b	17) a	18) c

Problem Set #2

1)

a) f '(x) = $12x^2$; F(x) = x^4 + C

b) f '(x) = 2 ; F(x) = x^2 + C

c) f '(x) = 0 ; F(x) = 5x + C

d) f '(x) = 14x + 6

F(x) = $\frac{7}{3}x^3 + 3x^2 - 2x$ + C

e) f '(x) = $-\frac{6}{x^4}$; F(x) = $-\frac{1}{x^2}$ + C

2)

a) $-\frac{32}{3} = -10\frac{2}{3}$

b) $-\frac{27}{4} = -6\frac{3}{4}$

3) $\lim\limits_{h\to\infty} \frac{f(x+h) - f(x)}{h}$

$\lim\limits_{h\to\infty} \frac{(x+h)^3 - 7(x+h) - (x^3 - 7x)}{h}$

$\lim\limits_{h\to\infty} \frac{x^3+3x^2h+3xh^2+h^3 - 7x - 7h - x^3 + 7x}{h}$

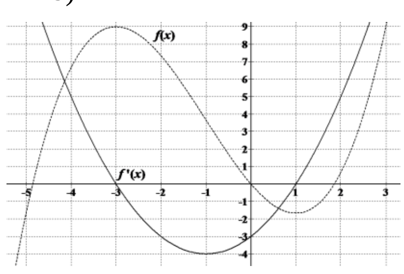
$\lim\limits_{h\to\infty} \frac{3x^2h+3xh^2+h^3 - 7h}{h}$

$\lim\limits_{h\to\infty} 3x^2 + 3xh + h^2 - 7$

therefore $f'(x) = 3x^2 - 7$

4) a) f '(x) = $x^2 + 2x - 3$

b)



4) c) The slope of f(x) is positive: (-∞, -3), (1, ∞). negative: (-3, 1) f '(x) has positive y-values when the slope of f(x) is positive. f '(x) has negative y-values when the slope of f(x) is negative.

4) d) Max: (-3, 9) Min: (1, $-\frac{5}{3}$)

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4) e) Min: (-1, -4). It's halfway between the local min and local max of f(x) on the x-axis. The graph is steepest between x equals –3 and 1. This is also where the graph changes from being concave down to becoming concave up. This is the *point of inflection*.

5) a   6) b   7) c   8) d   9) b   10) b
11) c   12) a   13) c   14) d   15) d   16) d

17) $f'(x) = \cos(x)$ **18)** $f'(x) = -\sin(x)$ **19)** Answers may vary.

20) a) $5x^4$ b) 3   c) $8x^3 - 2x$

21) **a)** $f'(x) = 0$ **b)** $f'(x) = 4x^3 - 12x^2 + 4x - 20$ **c)** $f'(x) = \frac{3x^2-4x-10}{(3x-2)^2}$

22) **a)** $\frac{dy}{dx} = 7x^6 - 35x^4 + 4x^3 - 24x^2 + 56$ **b)** $\frac{dy}{dx} = \frac{12x^6+29x^4+15x^2}{(4x^2+5)^2}$

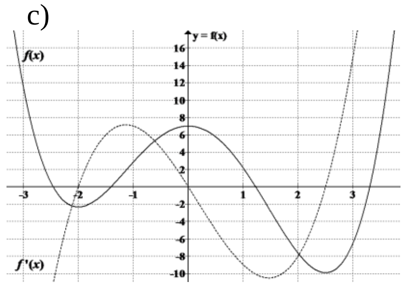
23) $f(x) = e^x$

Problem Set #3

1)
a) $f'(x) = 2x^3 - x^2 - 10x$

b) Mins: $(-2, -\frac{7}{3})$, $(\frac{5}{2}, -\frac{953}{96})$
Max: (0, 7)

c)



1) d) Positive slope:
$(-2, 0)$ and $(\frac{5}{2}, \infty)$.
Negative slope:
$(-\infty, -2)$ and $(0, \frac{5}{2})$

2)
a) Degrees: ≈ 0.017453
Radians:1

b) Degrees: 0
Radians: 0

c) 1

3) a)

$$f'(x) = \lim_{h\to 0} \frac{\cos(x+h) - \cos(x)}{h}$$

$$= \lim_{h\to 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h}$$

$$= \lim_{h\to 0} \frac{\cos(x)\cos(h) - \cos(x)}{h} - \lim_{h\to 0} \frac{\sin(x)\sin(h)}{h}$$

$$= \lim_{h\to 0} \cos(x)\frac{\cos(h)-1}{h} - \lim_{h\to 0} \sin(x)\frac{\sin(h)}{h}$$

$$= \cos(x)\cdot 0 - \sin(x)\cdot 1$$

$$\therefore \mathbf{f'(x) = -\sin(x)}$$

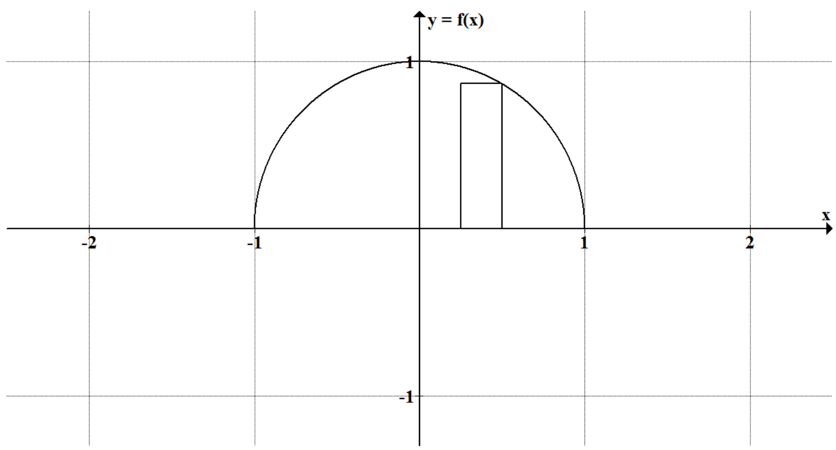
3) b)

$$f'(x) = \lim_{h\to 0}\frac{e^{x+h}-e^x}{h} = \lim_{h\to 0}\frac{e^x e^h - e^x}{h}$$

$$= \lim_{h\to 0} e^x\frac{e^h-1}{h} = e^x\cdot 1 \quad \therefore f'(x) = e^x$$

4)

a) The equation for a circle with a radius of one is $x^2 + y^2 = 1$ which means $y = \sqrt{1-x^2}$. Take the quarter circle in Quadrant I and make a really thin rectangle with its base on the *x*-axis that goes up to meet the curve. If we picture this rectangle revolving around the *x*-axis, a disk will be created with a height which is the width of the rectangle and a base with an area of πr^2.



The radius of the base of the disk is $\sqrt{1-x^2}$ and the height is *dx* (remember, the disk is really thin) which means the volume of the disk is $\pi(\sqrt{1-x^2})^2dx$. If we add up all of the disks from -1 to 1, we have our sphere:

$$\int_{-1}^{1}\pi(1-x^2)dx = \pi\int_{-1}^{1}(1-x^2)\,dx$$

$$= \pi\left(\frac{2}{3} - 0\right) = \frac{4}{3}\pi$$

b) Instead of 1, use *r* and you'll get $\frac{4}{3}\pi r^3$.

Calculus – Part I ANSWERS

Problem Set #4

1)

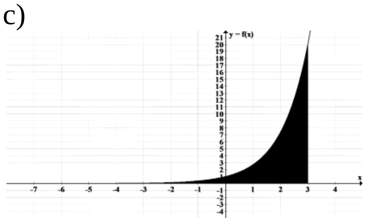
a) $\frac{\sqrt{3}}{2}$
b) 0
c) -1
d) -1
e) 0
f) 1
g) $e^2 \approx 7.4$
h) $\frac{1}{3}$

2)

a) 2
b) 1
c) 0
d) ln(5) – ln(2) ≈ 0.916
e) 1
f) $e^2 - 1 \approx 6.4$

3)

a) $e^3 \approx 20.1$
b) $e^3 \approx 20.1$
c)



$e^3 \approx 20.1$

4)

a) $3x^2 + 14x - 3\cos(x)$

b) $10\sin(x)\cos(x)$

c) $\frac{3\sin(x)}{\cos^4(x)}$

d) $e^x\cos(x) - e^x\sin(x)$

5)

a) $\frac{dy}{dx} = \sec^2 x$

b) $\frac{dy}{dx} = -\csc^2 x$

c) $\frac{dy}{dx} = \sec x \tan x$

d) $\frac{dy}{dx} = -\csc x \cot x$

e) $\frac{dy}{dx} = \frac{2x^3 - 12x^2}{(x-4)^2}$

6)

a) f '(x) = $5(\sin(x) + 3)^4(\cos(x))$
b) f '(x) = $5x^4\cos(x^5 + 3)$
c) f '(x) = $-\frac{5}{e^x}$
d) f '(x) = $3e^{3x}$
e) f '(x) = $-\frac{x}{\sqrt{1-x^2}}$
f) f '(x) = $-12x^3\sin(3x^4) - \sin(x)$
g) f '(x) = $-12x^3\sin(3x^4)\cos(x) - \sin(x)\cos(3x^4)$

h) f '(x) $f'(x) = \frac{1}{x}$

i) f '(x) = $\frac{1}{x}$

j) f '(x) = $\frac{1}{x}$

k) The derivative of the natural log of the product of a constant and a variable is equal to one over the variable.

Calculus – Part I ANSWERS

Problem Set #5

1)

a) $f'(x) = -\frac{1}{2}\sin(\frac{1}{2}x)$

b) $f'(x) = \frac{x\cos(x) - 2\sin(x)}{4x^3}$

c) $f'(x) = \cos^2(x) - \sin^2(x)$

d) $f'(x) = 2\sin(x)\cos^2(x) - \sin^3(x)$

e) $f'(x) = \frac{x^2 - 6x - 3}{(x-3)^2}$

f) $f'(x) = -\frac{5}{(x+3)^6}$

g) $f'(x) = -\frac{20}{(4x+3)^6}$

h) $f'(x) = 3\tan^2(x)\sec^2(x)$

2)

a) $f'(x) = -\frac{1}{2\sqrt{x^3}}$

b) $f'(x) = -\frac{7}{(2x-1)^2}$

c) $f'(x) = -3x^2\csc(x^3)\cot(x^3)$

d) $f'(x) = 4\cos(x) - 4x\sin(x)$

e) $f'(x) = -4\sin(4x)$

f) $f'(x) = -\frac{\sin(x)\cos(x)}{\sqrt{\cos^2(x)+1}}$

g) $f'(x) = \frac{1}{x}$

h) $f'(x) = 3x^2\ln(x) + x^2$

i) $f'(x) = \frac{1-3\ln(x)}{x^4}$

3)

a) $\frac{1}{2}$ d) $\frac{1}{5}$

b) 0 e) $\frac{1}{5}$

c) e f) $\frac{3}{2}$

4)

a) $\frac{2-\sqrt{2}}{2} \approx 0.293$

b) $\frac{\sqrt{3}-1}{2} \approx 0.366$

c) $\ln(10) \approx 2.3$

d) 1

e) e

5)

a) $f(x) = 5^x$

$f(x) = e^{(\ln 5)x}$

$f'(x) = (\ln 5)\, e^{(\ln 5)x}$

$f'(x) = (\ln 5)5^x$

b) $f(x) = 5^x$

$f(x) = e^{\ln 5^x}$

$f(x) = e^{(\ln 5)x}$

$F(x) = \int e^{(\ln 5)x}dx = \frac{1}{\ln 5}e^{(\ln 5)x}$

$F(x) = \frac{1}{\ln 5}5^x$

6)

a) $\frac{dy}{dx} = (\ln a)a^x$

b) $F(x)\ \frac{1}{\ln a}a^x$

7) $\log_2 x = \frac{\ln x}{\ln 2}$

Thus: $\frac{d}{dx}\log_2 x = \frac{1}{x\ln 2}$

8) a) The anti-derivative, F(x) = ln x

$$\text{Area} = \int_1^{\infty} \frac{1}{x}\,dx = \Big|_1^{\infty} \ln x = \ln(\infty) - \ln(1) = \infty - 0 = \infty$$

b) The vortex is created by rotating the above area about the x-axis? We imagine taking vertical slices of the vortex. The volume is then the sum of the infinitely many and progressively smaller (and infinitely thin) disks.

The volume of a single disk is $\pi[f(x)]^2\,dx$.

The volume of the entire vortex is:

$$\int_1^{\infty} \pi[f(x)]^2\,dx = \int_1^{\infty} \pi x^{-2}\,dx = \Big|_1^{\infty} -\pi x^{-1} = \left(-\tfrac{\pi}{\infty}\right) - \left(-\tfrac{\pi}{1}\right) = \boldsymbol{\pi}$$

The volume is just π even though the region had an infinite area!

9) a) The anti-derivative, $F(x) = -\frac{1}{x}$

$$\text{Area} = \int_1^{\infty} \frac{1}{x^2}\,dx = \Big|_1^{\infty} -\frac{1}{x} = \left(-\tfrac{1}{\infty}\right) - \left(-\tfrac{1}{1}\right) = \mathbf{1}$$

Surprisingly, the area is 1 even when the region is infinitely long.

b) We use the same method as with the previous volume problem.

The volume of a single disk is $\pi[f(x)]^2\,dx = \pi x^{-4}\,dx$.

The volume of the entire vortex is:

$$\int_1^{\infty} \pi x^{-4}\,dx = \Big|_1^{\infty} -\tfrac{1}{3}\pi x^{-3} = \left(-\tfrac{\pi}{3\infty^3}\right) - \left(-\tfrac{\pi}{3\cdot 1^3}\right) = \boldsymbol{\frac{\pi}{3}}$$

Calculus – Part I ANSWERS

Problem Set #6

1)

a) $-\frac{\sqrt{2}}{2}$ e) $\frac{1}{3}$

b) 1 f) $\frac{1}{3}$

c) $-\frac{1}{2}$ g) $\frac{4}{3}$

d) $\sqrt{e}$ h) $\frac{\ln 3}{9}$

2)

a) $\frac{1}{2}$

b) 0

c) $-\ln(0.1) = \ln(10) \approx 2.30$

d) $\frac{e^4 - 1}{e^2} \approx 7.25$

3)

a) $5\cos(x)(\sin(x) + 3)^4$

b) $72(4x - 1)^5$

4)

a) $\frac{dy}{dx} = \frac{x\cos(x) - \sin(x)}{x^2}$

b) $\frac{dy}{dx} = \frac{2x\sin(x)\cos(x) - 3\sin^2(x)}{x^4}$

c) $\frac{dy}{dx} = \frac{2(x+3)e^{2x} - 5e^{2x}}{(x+3)^6}$

d) $\sin(\frac{\pi}{2} - x) = \cos(x)$

This is because the original problem is equal to $\sin(x)$

e) $3x^2\sin(x) + x^3\cos(x)$

f) $3x^2\sin\left(\frac{x}{3}\right) + \frac{1}{3}x^3\cos\left(\frac{x}{3}\right)$

5)

a) $f'(x) = 2\sec(2x)\tan(2x)$

b) $f'(x) = \cot(x)$

c) $f'(x) = \frac{\cos(\ln x)}{x}$

d) $f'(x) = 4\sin(x)\cos(x)$

e) $f'(x) = 2\sin(x)\cos^3(x) - 2\cos(x)\sin^3(x)$

f) $f'(x) = \frac{5\cos\sqrt{5x+3}}{2\sqrt{5x+3}}$

g) $f'(x) = 4x^3e^x + x^4e^x$

h) $f'(x) = \frac{e^x\sqrt{x} - \frac{e^x}{2\sqrt{x}}}{x}$

$f'(x) = \frac{\frac{2x\,e^x}{2\sqrt{x}} - \frac{e^x}{2\sqrt{x}}}{\frac{x}{1}}$

$f'(x) = \frac{(2x-1)\,e^x}{2\sqrt{x}} \bullet \frac{1}{x}$

$f'(x) = \frac{(2x-1)\sqrt{x}\,e^x}{2x^2}$

6) The function in graph *a* is the derivative of the function in graph *e*.
b is the derivative of *f*.
c is the derivative of *h*.
d is the derivative of *g*.
i is the derivative of *j*.

7) The function in graph *i* is: $-\sin(x)$. The function in graph *j* is: $3 + \cos(x)$

Calculus – Part II ANSWERS

Problem Set #1

1)

a) $\frac{dy}{dx} = 28x^6 - 6x + 1$

b) $\frac{dy}{dx} = 1 + \sin x$

c) $\frac{dy}{dx} = 2\cos x - \sec^2 x$

d) $\frac{dy}{dx} =$

$2x\sin x + x^2 \cos x$

e) $\frac{dy}{dx} = -8x \sin(4x^2)$

f) $\frac{dy}{dx} = 3e^{3x}$

g) $\frac{dy}{dx} = 4\tan x \sec x$

h) $\frac{dy}{dx} = \frac{-1}{1+\sin x}$

i) $\frac{dy}{dx} = 2\cot x$

2)

a) $\frac{dy}{dx} = \frac{2x+3}{2y}$

b) $\frac{dy}{dx} = \frac{7y}{2y-7x}$

c) $\frac{dy}{dx} = -\frac{2x}{\sin y} = -2x \csc y$

d) $\frac{dy}{dx} = -\frac{x}{y}$

e) $\frac{dy}{dx} = \frac{3-3x^2}{10y-2}$

f) $\frac{dy}{dx} = \frac{e^x}{3y^2+2y}$

g) $\frac{dy}{dx} = -\frac{\ln x+1}{\ln y+1}$

h) $\frac{dy}{dx} = \frac{3y-x}{9y-3x-1}$

3) $\frac{dA}{dt} = 3$

$A(r) = \pi r^2$

$A'(r) = \frac{dA}{dr} = 2\pi r$

When $r = 7$, $\frac{dA}{dr} = 14\pi$

$\frac{dr}{dt} = \frac{dA}{dt} \div \frac{dA}{dr} =$

$\frac{3}{14\pi} \approx 0.0682$ ft/sec.

Problem Set #2

1)

a) $\frac{dy}{dx} = \frac{15x^2 - 7}{6y}$

b) $\frac{dy}{dx} = \frac{-2xy - y^2}{2xy + x^2}$

c) $\frac{dy}{dx} = \frac{2-\cos x}{2y}$

d) $\frac{dy}{dx} = \frac{2x}{2-\cos y}$

e) $\frac{dy}{dx} = \frac{1 - \ln x}{x^2}$

f) $\frac{dy}{dx} = e^x\cos x - e^x\sin x$

g) $\frac{dy}{dx} =$

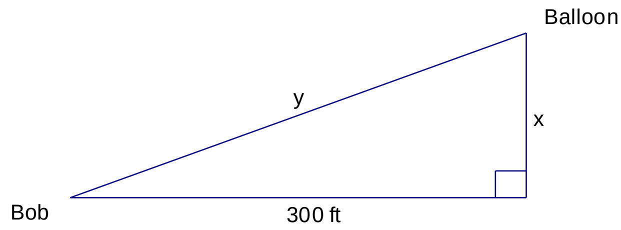
$4x^3 \sec x + x^4\tan x \sec x$

h) $\frac{dy}{dx} = \frac{y \sin x - \sin y}{x\cos y + \cos x}$

2) $\frac{1}{2\pi} \approx 0.159$ cm/sec

Calculus – Part II ANSWERS

3)



$\frac{dx}{dt}$ = 140 ft/min. $y^2 = 300^2 + x^2$

This problem can be solved in two ways:

<u>Without Implicit Differentiation</u>

$y = \sqrt{300^2 + x^2} = (300^2 + x^2)^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}(300^2 + x^2)^{-\frac{1}{2}} \cdot 2x = \frac{x}{\sqrt{300^2+x^2}}$

$\frac{dy}{dt} = \frac{dx}{dt} \cdot \frac{dy}{dx} = \frac{140x}{\sqrt{300^2+x^2}}$

When x = 90 ft, $\frac{dy}{dt} \approx$ 40.2 ft/min.

<u>With Implicit Differentiation</u>

$2y\frac{dy}{dx} = 0 + 2x\frac{dx}{dx}$

$\frac{dy}{dx} = \frac{x}{y}$

$\frac{dy}{dt} = \frac{dx}{dt} \cdot \frac{dy}{dx} = \frac{140x}{y}$

When x = 90 ft., $y \approx$ 313.21 ft., $\frac{dy}{dt} \approx$ 40.2 ft/min.

4) $1\frac{dx}{dx} = e^y \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{e^y}$

Since $x = e^y$

$\frac{dy}{dx} = \frac{1}{x}$

5)

a) $v(t) = 8t$
Starts at rest.
$d(4) = 64$ m
$v(4) = 32$ m/sec.

b) $v(t) = 6t + 6$
Starts by being thrown down at 6 m/s.
$d(2) = 24$ m
$v(2) = 18$ m/sec.

5) c) $v(t) = 4t – 6$
Starts by being thrown up the plane at 6 m/s.
$d(3) = 0$ m
$v(3) = 6$ m/sec.

6)

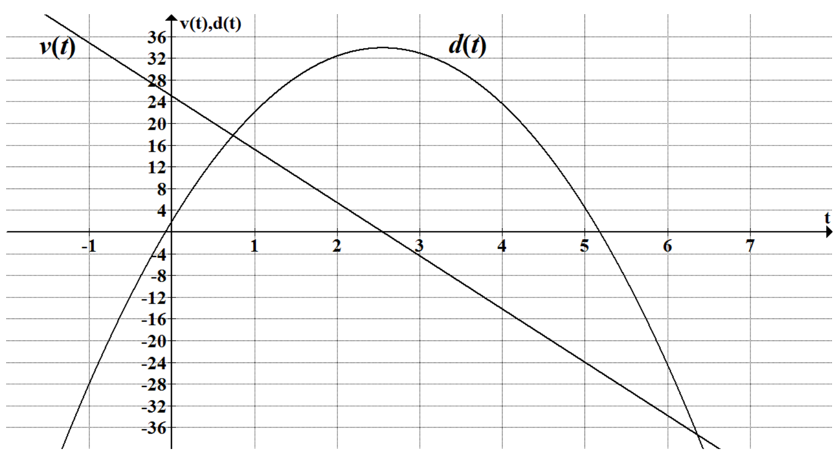
a) $v(t) = -9.8t + 25$

b) $d(t) = -4.9t^2 + 25t + 2$

c) $d(1) = 22.1$ m
$d(3) = 32.9$ m
$d(6) = -24.4$ m so the ball's height is 0.
$v(1) = 15.2$ m/sec
$v(3) = -4.4$ m/sec
$v(6) = -33.8$ m/sec

d) ≈ 33.89 m

6) e)



6) f) The velocity stays positive while the ball's distance from the ground increases. The velocity becomes negative as the ball heads back towards to ground. The velocity is zero when the ball is at its maximum distance from the ground.

Calculus – Part II ANSWERS

7)

a) Using implicit differentiation we get:

$\frac{dy}{dx} = \frac{3y - x^2}{y^2 - 3x}$ Thus:

At (2,4), the slope is : $\frac{4}{5}$

At (4,2), the slope is: $\frac{5}{4}$

b) Using $x = 2$, we get

$y^3 - 18y + 8 = 0$

By using polynomial long division, this becomes

$(y - 4)(y^2 + 4y - 2) = 0$.

Thus $y = -2 \pm \sqrt{6}$

The coordinates are approximately:

(2, 0.45), (2, -4.45)

c) $x = -2 \pm \sqrt{6}$

(0.45, 2), (-4.45, 2)

d) $y = -1 \pm \sqrt{33}$

(4, 4.74), (4, -6.74)

e) $x = -1 \pm \sqrt{33}$

(4.74, 4), (-6.74, 4)

f) $\frac{dy}{dx} = \frac{x^2 - 3y}{3x - y^2} = -1$

$x^2 - 3y = y^2 - 3x$

Thus $y = x$

Therefore using

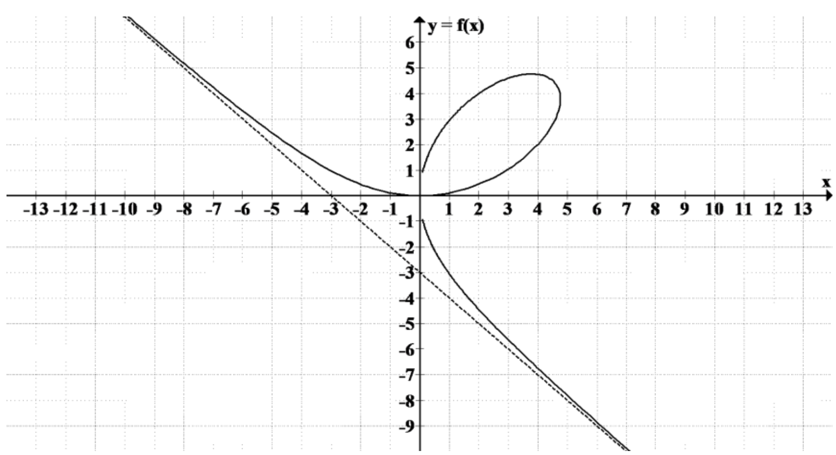
$x^3 + y^3 = 9xy$

yields $2x^3 = 9x^2$

Thus the answer is:

$\left(\frac{9}{2}, \frac{9}{2}\right)$

7) g)



Calculus – Part II ANSWERS

Problem Set #3

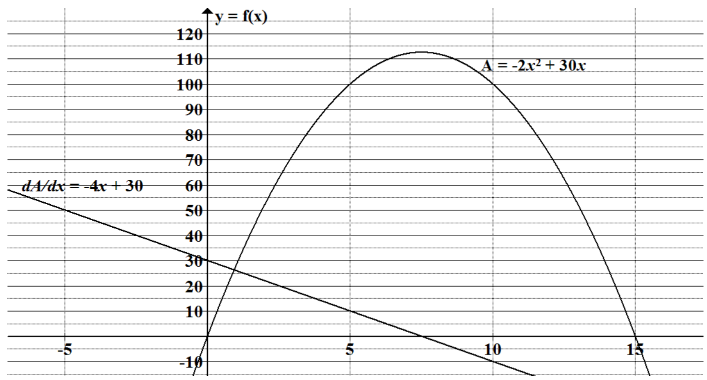
1)

a) $y = -2x + 30$

b) $A = -2x^2 + 30x$

c) $\frac{dA}{dx} = -4x + 30$. This is the rate at which the area of the garden grows with respect to the width.

d)



When A is increasing, $\frac{dA}{dx}$ is positive. At A's maximum, $\frac{dA}{dx} = 0$. When A is decreasing, $\frac{dA}{dx}$ is negative.

1) e) $x = 7.5$ ft, $y = 15$ ft.

2)

a) $V = h\pi r^2$

b) Since $r^2 = 1 - \frac{1}{4}h^2$

$V(h) = -\frac{\pi}{4}h^3\pi + \pi h$

c) $\frac{dV}{dh} = V'(h) = -\frac{3\pi}{4}h^2 + \pi$

d) The rate of change of the volume of the cylinder with respect to its height.

2) e) $V(0.5) = \frac{15\pi}{32} \approx 1.4726$
When the height of the cylinder is 0.5, the volume is ≈ 1.4726.

f) $V'(0.5) = \frac{13\pi}{16} \approx 2.5525$
The volume of the cylinder is increasing at a rate of ≈ 2.5525 when the height is 0.5.

g) $V(1.5) = \frac{21\pi}{32} \approx 2.0617$
When the height of the cylinder is 1.5, the volume is ≈ 2.0617.

2) cont'd

h) $V'(1.5) = -\frac{11\pi}{16}$

≈ -2.16

The volume of the cylinder is decreasing at a rate of ≈ -2.16 when the height is 1.5.

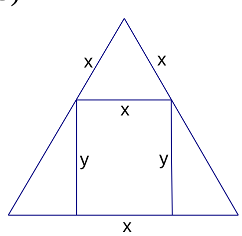
i) $h = \frac{2\sqrt{3}}{3} \approx 1.1547$

$r = \frac{\sqrt{6}}{3} \approx 0.8165$

$V \approx 2.4184$

j) The maximum of $V(h)$ is a zero of $V'(h)$.

3)



Let x be the width and y be the height of the rectangle. If A is the area of the rectangle then $A = xy$. The height of the smaller equilateral triangle with edges of length x is both $\frac{\sqrt{3}}{2}x$ and $\frac{\sqrt{3}}{2} - y$ thus

$\sqrt{3}x = \sqrt{3} - y.$

So $y = \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}x$

Thus $A = -\frac{\sqrt{3}}{2}x^2 + \frac{\sqrt{3}}{2}x$, and…

$\frac{dA}{dx} = -\sqrt{3}x + \frac{\sqrt{3}}{2}$

Solving for x:

$-\sqrt{3}x + \frac{\sqrt{3}}{2} = 0$

$x = \frac{1}{2}$

Finally, the dimensions of the rectangle at its largest possible area are: $\frac{1}{2}$ x $\frac{\sqrt{3}}{4}$

4) $\frac{dV}{dt} = 30$

$r{:}h = 8{:}12 = 2{:}3$

Thus $r = \frac{2}{3}h$

$V = \frac{1}{3}h\pi r^2$

$V = \frac{1}{3}h\pi\left(\frac{2}{3}h\right)^2$

$V = \frac{4\pi}{27}h^3$

$\frac{dV}{dh} = \frac{4\pi}{9}h^2$

$\frac{dh}{dt} = \frac{dV}{dt} \div \frac{dV}{dh} = \frac{135}{2\pi h^2}$

a) $\frac{dh}{dt} = \frac{135}{8\pi} \approx$

5.37 cm/sec.

b) $\frac{dh}{dt} = \frac{27}{40\pi} \approx$

0.215 cm/sec

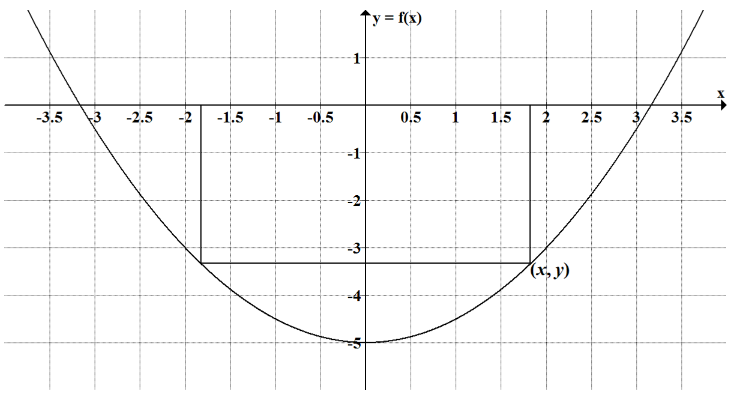
c) 26.81 seconds.

Calculus – Part II ANSWERS

Problem Set #4

1) a) $\frac{y-4}{2y-x}$ b) $\frac{1-2xy}{x^2}$ c) $\sec y$ d) $-10x\sin(5x^2)$

2)



$A = 2x(-y)$ and $y = \frac{1}{2}x^2 - 5$ thus $A = -x^3 + 10x$.

$\frac{dA}{dx} = -3x^2 + 10$. Setting $\frac{dA}{dx}$ equal to 0 gives us

$x = \sqrt{\frac{10}{3}} \approx 1.825741$. Thus the dimensions of the rectangle are:

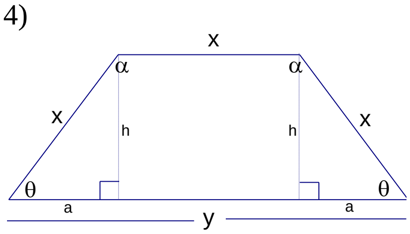
$\frac{2\sqrt{30}}{3} \text{ x } \frac{10}{3}$ or $\approx 3.651482 \text{ x } 3\frac{1}{3}$

3) Using the volume of a cylinder, we get $500 = \pi hr^2$. Thus $h = \frac{500}{\pi r^2}$
The surface area of a cone (including the top and the bottom) is
$A = 2\pi hr + 2\pi r^2 \quad \rightarrow \quad A = \left(\frac{500}{\pi r^2}\right)2\pi r + 2\pi r^2 \quad \rightarrow \quad A = \frac{1000}{r} + 2\pi r^2$
$\frac{dA}{dr} = 4\pi r - \frac{1000}{r^2}$ Setting $\frac{dA}{dr}$ equal to 0 and solving for r gives:

$r = 5\sqrt[3]{\frac{2}{\pi}} \approx 4.30$ and $h \approx 8.6$

Calculus – Part II ANSWERS

4)



Because $y = x + 2a$ and
$a = x \cos\theta$, $y = x + 2x\cos\theta$.
$h = x \sin\theta$
Let A be the area of a trapezoid.
Thus $A = \frac{1}{2}(x + y)\,h$

$A = \frac{1}{2}(x + x + 2x\cos\theta)\,x\sin\theta$

$A = x^2 \sin\theta + x\cos\theta\sin\theta$

Let $x = 1$, thus

$A = \sin\theta + \cos\theta\sin\theta$

$\frac{dA}{d\theta} = \cos\theta - \sin^2\theta + \cos^2\theta$

Since $\sin^2\theta = 1 - \cos^2\theta$

$\frac{dA}{d\theta} = 2\cos^2\theta + \cos\theta - 1$

Set $\frac{dA}{d\theta} = 0$ and let $u = \cos\theta$.

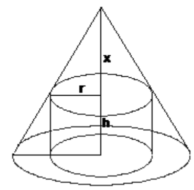
This $2u^2 + u - 1 = 0$

$u = \cos\theta = -1, \frac{1}{2}$.

Because $\theta \neq 180°$, $\cos\theta = \frac{1}{2}$

Thus $\theta = 60°$ and $\alpha = 120°$.

5)



$x + h = 16$ so $x = 16 - h$
$x : r = 16 : 10$ so $x = \frac{8}{5}r$.
$\frac{8}{5}r = 16 - h$
Thus $h = 16 - \frac{8}{5}r$
Let V be the volume of the cylinder. Thus $V = \pi r^2 h$.
Substituting h give us:

$V = 16\pi r^2 - \frac{8}{5}\pi r^3$

$\frac{dV}{dr} = 32\pi r - \frac{24}{5}\pi r^2$

Setting $\frac{dV}{dr} = 0$ gives us:

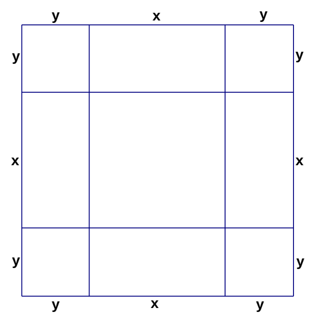
$r = \frac{20}{3} = 6\frac{2}{3}$ inches

$h = \frac{16}{3} = 5\frac{1}{3}$ inches

$V = \frac{6400}{27}\pi$

Problem Set #5

1)



Let V be the volume of the box. Therefore $V = x^2y$.

Since $2y + x = 20$, $x = 20 - 2y$.

$V = (20 - 2y)^2y = 4y^3 - 80y^2 + 400y$.

$\frac{dV}{dy} = 12y^2 - 160y + 400$.

Setting $\frac{dV}{dy} = 0$ gives $y = 10, \frac{10}{3}$

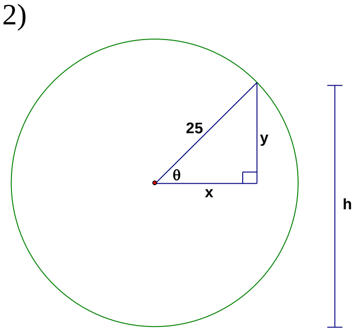
If $y = 10$, then $x = 0$ which is not possible. Therefore

$y = \frac{10}{3} = 3\frac{1}{3}$ and $x = \frac{40}{3} = 13\frac{1}{3}$.

The largest possible volume is

$\frac{16000}{27} = 592\frac{16}{27} \approx 592.6 \text{ cm}^3$

2)



$\frac{d\theta}{dt} = 8\pi$ per minute. Let h be the height off the ground of the passenger. Therefore

$h = y + 25$ and $y = 25\sin\theta$

$h = 25\sin\theta + 25$.

The rate at which h is changing with respect to θ is $\frac{dh}{d\theta} = 25\cos\theta$.

Let $v(t)$ be the velocity of the passenger in feet/min.

$v(t) = \frac{dh}{dt} = \frac{dh}{d\theta} \cdot \frac{d\theta}{dt}$

$= (25\cos\theta)(8\pi) = 200\pi\cos\theta$

a) When a passenger is $\frac{1}{8}$ turn from the top, $\theta = \frac{\pi}{4}$.

$200\pi\cos\left(\frac{\pi}{4}\right) = 100\sqrt{2}\pi$

≈ 444 ft/min or 5.05 mph.

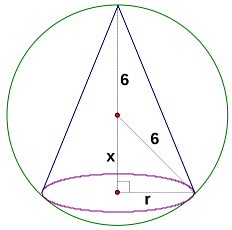
b) Velocity is maximized when the derivative (acceleration) is zero.

$\frac{dv}{dt} = -200\pi\sin\theta$.

Setting $\frac{dv}{dt} = 0$ gives $\theta = 0°$ or $180°$. The Ferris wheel is heading up when $\theta = 0°$ so using $\theta = 180°$ gives $v(t) = 200\pi$

≈ 628 ft/min or 7.14 mph.

3)



Let V be the volume of the cone. Thus $V = \frac{1}{3}\pi r^2 h$.

$6 + x = h$ and $r^2 = 6^2 - x^2$

$V = \frac{1}{3}\pi(6^2 - x^2)(6 + x)$

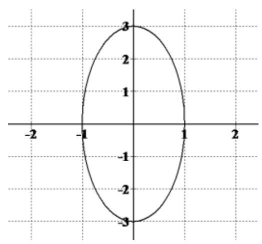
$V = -\frac{1}{3}\pi(x^3 + 6x^2 - 36x - 216)$

$\frac{dV}{dx} = -\frac{1}{3}\pi(3x^2 + 12x - 36)$

Setting $\frac{dV}{dx} = 0$ yields $x = -6, 2$.

x must be positive so $h = 8$ cm and $r = 4\sqrt{2} \approx 5.657$ cm.

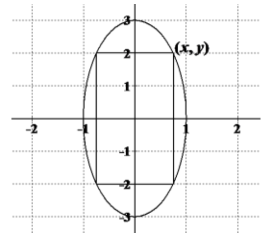
4)
a)



b) $\frac{dy}{dx} = -\frac{9x}{y}$

c) When $y = 2$, $x = \frac{\sqrt{5}}{3}$ and the slope is $-\frac{3\sqrt{5}}{2}$

4) d)



$9x^2 + y^2 = 9 \rightarrow y = 3\sqrt{1 - x^2}$

Let A be the area of the rectangle. Thus $A = 4xy$.

Therefore $A = 12x\sqrt{1 - x^2}$

$A = 12\sqrt{x^2}\sqrt{1 - x^2}$

$A = 12\sqrt{x^2 - x^4}$

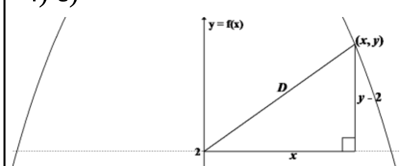
$\frac{dA}{dx} = \frac{12(1 - 2x^2)}{\sqrt{1 - x^2}}$.

Setting $\frac{dA}{dx} = 0$, yields

$x = \frac{\sqrt{2}}{2}$, $y = \frac{3\sqrt{2}}{2}$.

Thus the dimensions of the rectangle are: $\sqrt{2}$ x $3\sqrt{2}$

4) e)



$x^2 = \frac{9 - y^2}{9}$ and $D^2 = x^2 + (y - 2)^2$

thus $D = \sqrt{\frac{8}{9}y^2 - 4y + 5}$

$\frac{dD}{dy} = \frac{\frac{8}{9}y - 2}{\sqrt{\frac{8}{9}y^2 - 4y + 5}}$. Setting $\frac{dD}{dy} = 0$

yields $y = \frac{9}{4}$, $x = \frac{\sqrt{7}}{4}$

Therefore $\boldsymbol{D = \frac{\sqrt{2}}{2}}$

Calculus – Part II ANSWERS

Problem Set #6

1) a) $\frac{dy}{dx} = \frac{y\sin x - \sin y}{\cos x + x\cos\, y}$

b) $\frac{dy}{dx} = 15x^4 - 3x^2$

c) $\frac{dy}{dx} = \frac{15x^2y^2}{1-10x^3y}$

d) $\frac{dy}{dx} = -\frac{x^2}{y^2}$

e) $\frac{dy}{dx} = \frac{\cos^2(y^3)}{3xy^2}$

2) $y = \frac{3}{4}x - \frac{25}{4}$

3) $\frac{dV}{dt} = 10$ cm^3/sec

Let V be the volume of the cone. Then $V = \frac{1}{3}\pi r^2 h$.

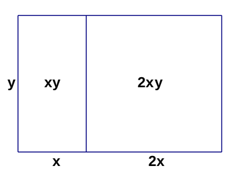
$r = \frac{3}{4}h$ thus $V = \frac{3}{16}\pi h^3$

$\frac{dV}{dh} = \frac{9}{16}\pi h^2$.

$\frac{dh}{dt} = \frac{dV}{dt} \div \frac{dV}{dh} = \frac{160}{9\pi h^2}$.

When $h = 30$, $\frac{dh}{dt} = \frac{8}{405\pi} \approx$ 0.00629 cm/sec

4)



Let $A = 3xy$ (the area of the entire plot of land).

$3y + 6x = 500$

$y = -2x + \frac{500}{3}$

$A = -6x^2 + 500x$, and…

4) (cont'd) Therefore…

$\frac{dA}{dx} = -12x + 500$

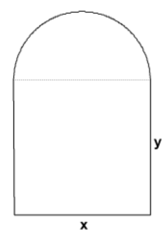
Setting $\frac{dA}{dx} = 0$ gives

$x = \frac{125}{3} = 41\frac{2}{3}$ ft.

Thus the dimensions of the two plots of land are:

$41\frac{2}{3}$ ft x $83\frac{1}{3}$ ft, $83\frac{1}{3}$ ft x $83\frac{1}{3}$ ft

5)



Let A be the area.

Thus $A = xy + \frac{1}{2}\pi(\frac{1}{2}x)^2$

$A = xy + \frac{\pi}{8}x^2$

Perimeter: $x + 2y + \frac{\pi}{2}x = 12$

$y = -\frac{1}{2}x - \frac{\pi}{4}x + 6$

Subbing for y gives:

$A = -\frac{1}{2}x^2 - \frac{\pi}{8}x^2 + 6x$

$\frac{dA}{dx} = -x - \frac{\pi}{4}x + 6$

Setting $\frac{dA}{dx} = 0$ yields

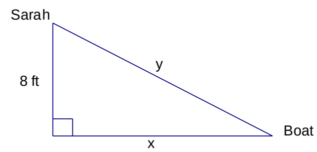
$x = \frac{6}{1+\frac{\pi}{4}} \approx 3.361$

$y = \frac{3}{1+\frac{\pi}{4}} \approx 1.6803$

Calculus – Part II ANSWERS

6)

a) The boat speeds up increasingly quickly as it approaches the dock.



$y^2 = 8^2 + x^2 \qquad \frac{dy}{dt} = 3$

Finding a formula for $\frac{dx}{dt}$

Without implicit differentiation:

$y = \sqrt{8^2 + x^2} = (8^2 + x^2)^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}(8^2 + x^2)^{-\frac{1}{2}}(2x)$

$= \frac{x}{\sqrt{8^2+x^2}}$

$\frac{dx}{dt} = \frac{dy}{dt} \div \frac{dy}{dx} = 3 \div \frac{x}{\sqrt{8^2+x^2}}$

$\frac{dx}{dt} = \frac{3\sqrt{8^2+x^2}}{x} \quad \left(= \frac{3y}{x}\right)$

With implicit differentiation:

$y^2 = 8^2 + x^2$

$2y\frac{dy}{dx} = 0 + 2x\frac{dx}{dx}$

$\frac{dy}{dx} = \frac{x}{y}$

$\frac{dx}{dt} = \frac{dy}{dt} \div \frac{dy}{dx} = 3 \div \frac{x}{y}$

Therefore $\frac{dx}{dt} = \frac{3y}{x}$

6) b) If $y = 120$ ft, then

$x \approx 119.733$ ft.

$\frac{dx}{dt} \approx 3.0067$ ft/sec.

6) c) If $y = 15$ ft, then

$x \approx 12.689$ ft.

$\frac{dx}{dt} \approx 3.5465$ ft/sec.

7)

$2h + 2w = 6 \rightarrow w = 3 - h$

$2h + L = 4 \rightarrow L = 4 - 2h$

$V = h\,w\,L = h(3 - h)(4 - 2h)$

$V = 2h^3 - 10h^2 + 12h$

$\frac{dV}{dh} = 6h^2 - 20h + 12$

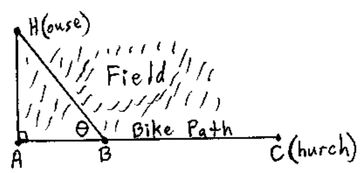
Setting $\frac{dV}{dh} = 0$ yields

$h \approx 2.55$ <u>or</u> 0.785.

2.55 is too big.

Therefore:
$h \approx 0.785$ inches
$w \approx 2.22$ inches
$L \approx 2.43$ inches

8)



AH = 3; AC = 5
BH = 3/sin θ; AB = 3/tan θ

$BC = 5 - \frac{3}{\tan\theta} = 5 - \frac{3\cos\theta}{\sin\theta}$

d = BC + BH ; t = d/r

$t = \left(\frac{5}{25} - \frac{3\cos\theta}{25\sin\theta}\right) + \frac{3}{10\sin\theta}$

$t = \frac{5}{25} - \frac{3}{25}\cot\theta + \frac{3}{10}\csc\theta$

$\frac{dt}{d\theta} = \frac{3}{25}\csc^2\theta - \frac{3}{10}\csc\theta\cot\theta$

setting to zero, gives us:

$\cos\theta = \frac{10}{25} \rightarrow \theta \approx 66.4°$

Plugging in above gives:
t ≈ .475 hrs or 28.5 minutes