Answers for Grade 11 Group Assignments - Quarter #4

Notes:

- Answers for group assignment problems that are out of the workbook can be found in the "G11 – Workbook Answer Key".
- This answer key doesn't include all answers.

Week 25

- Isa's Birthday Puzzle.
  - We assume that both Ning and John are being truthful and not making any mistakes in their thinking.
  - ***What Ning says first***. Because Ning knows only the month, and at first he says "John doesn't know either", then we know that Ning was told either July or August. This is because if he had been told either May or June, then it would be possible for John to have been told either 18 or 19, which are unique days, and in that case John would have immediately known Isa's birthday. So, we can eliminate May and June, and Ning must have been told either "July" or "August".
  - ***What John says***. John knows how Ning is thinking. When John says: "I didn't know originally, but now I do.", Ning knew that 14 isn't the day, because if that had been the case, John could still not know whether the month was July or August. So we know John was told either 15, 16, or 17.
  - ***What Ning says next***. How can it be that after John announces that he now knows Isa's birthday, that Ning could also know for certain? Well, if Ning had been told "August", then he still wouldn't be able to decide between the 15^{th} and the 17^{th}. Therefore, we now know that Ning must have been told July, because the only possibility for a remaining date (again, 14 has been eliminated) is 16.
  - Isa's birthday is **July 16**.
- The A-C Train Problem.
  - One effective strategy is to start with shorter trains and build up. There is 1 way to build a train of length 0, 1 way to build a train of length 1, 2 ways to build a train of length 2, etc. Let x_n be the number of possible trains having a length of n. The sequence (starting with n=0) is then x_n = 1,1,2,3,5,8,13… We hopefully notice that the value of any x_n is simply the sum of the two previous terms ($x_n = x_{n-1} + x_{n-2}$). This is the Fibonacci sequence. Continuing in this way we get achieve a result of 233 possible trains having a length of 12, and there are 10,946 possible trains having a length of 20.

Week 26

1) x_n = 1, 1, 3, 5, 11, 21, 43, 85, 171, 341, 683, 1365, 2731
2) $x_n = x_{n-1} + 2\,x_{n-2}$
3) $0.35 \cdot 68 + 0.65 \cdot 90 = 82.3$
4) $(^2/_{11})(85) + (^3/_{11})(68) + (^6/_{11})(90) \approx 83.1$
5) $(^5/_7)(80) + (^2/_7)(68) \approx 76.57$kg
6) 63
7) 190
8) 376
9) $x^3 + x^2 + x + 1$
10) $\frac{4^7 - 1}{4 - 1} = 5461$
11) $\frac{3^{19} - 1}{3 - 1} = 581{,}130{,}733$
12) Same as #10
13) Same as #11
14) $\frac{3^{n+1} - 1}{2}$
15) recursive: $x_n = 3x_{n-1} - 4$;
    general: $x_n = 5 \cdot 3^n + 2$
16) $x_n = \frac{(ax_0 - x_0 + b)a^n - b}{a - 1}$
17) $\Phi = \frac{1 + \sqrt{5}}{2} \approx 1.61803398874989485$
    $\hat{\Phi} = \frac{1 - \sqrt{5}}{2} \approx -0.61803398874989485$
18) 1, 2, 1.5, $1.\overline{6}$, 1.6, 1.625… It gets closer and closer to Φ.
19) a) $\Phi + 1$ b) $\hat{\Phi} + 1$ c) -1 d) $\hat{\Phi}$
    e) $-\hat{\Phi}$ f) Φ g) 1 h) $\sqrt{5}$

Week 27

- 1) 5   2) $\frac{1}{5}$ 3) 25   4) $\frac{1}{25}$ 5) 4   6) $\frac{1}{16}$ 7) -1   8) 2   9) 0   10) $\frac{1}{4}$
- *The Dartboard Problem (Part I).*

  a) $\frac{1}{12}$ or $\approx 8.33\%$ b) $\frac{11}{12}\cdot\frac{1}{12} \rightarrow \frac{11}{144}$ or $\approx 7.64\%$ c) $\left(\frac{11}{12}\right)^2\cdot\frac{1}{12} \rightarrow \approx 7.00\%$

  d) $\approx 23.0\%$ e) $\approx 77.0\%$ f) $\left(\frac{11}{12}\right)^{19}\cdot\frac{1}{12} \rightarrow \approx 1.60\%$

  g) This is a weighted average problem. $15(0.2) + 32(0.3) + 24(0.5) \rightarrow$ 24.6

Week 28

- *The Dartboard Problem (Part II).* I will go over the solution in Wednesday's lecture (Lecture #2).
- *Diophantus's Riddle.* If x is the length of Diophantus's life, and y is the length of his son's life, then we get the equations: $x = \frac{1}{6}x + \frac{1}{12}x + \frac{1}{7}x + 5 + y + 4$ and $y = \frac{1}{2}x$

  Solving for x tells us that Diophantus lived 84 years.
- *Two-Digit Numbers.*

  a) The number is 18.

  b) Surprisingly, it turns out that each number simply needs to be such that one digit is twice the other digit. The possible answers are 12 & 21, 24 & 42, 36 & 63, and 48 and 84.

  c) The numbers are 41 and 51.
- *Squares and Circles.*

  The smaller square has a length of $\frac{3}{5}$. The radii are $\frac{39}{320}$ and $\frac{1}{16}$.

  These solutions can be arrived at in the following way. The left-most drawing (shown below) shows how we can find the dimensions of the smaller square. We then get the two equations:

  $$x + 2y = 1 \qquad 1^2 = x^2 + (x + y)^2$$

  Solving these two equations leads to the answer $x = \frac{3}{5}$.

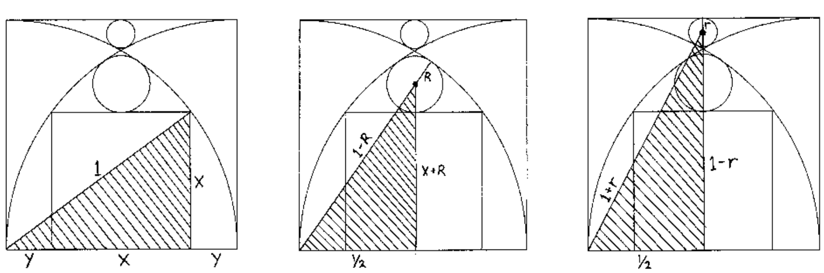
  The drawing in the center is based on the fact that a line drawn from the point of tangency (of the larger circle and the quarter-arc) to the bottom-left corner of the larger square must pass through the center of the larger circle. This leads to the equation:

  $$(1 - R)^2 = (\tfrac{1}{2})^2 + (x + R)^2$$

  Knowing that x (the length of the smaller square) is equal to $\frac{3}{5}$ leads to an answer of $R = \frac{39}{320}$.

  The drawing at the right leads to the equation:

  $(1 + r)^2 = (\tfrac{1}{2})^2 + (1 - r)^2$, and the answer of $r = \frac{1}{16}$.



Week 29

- *Cell Phone Decisions*.
  This is a good exercise to show a practical application of Cartesian geometry. So, I we graph all three equations and *see* the results. The x-axis ought to be the number of cell phone minutes and the y-axis ought to be the total cost for the particular plan. The end result is that the break-even point between the first two plans is ≈494 minutes per month, and the break-even point between the last two plans is 1000 minutes per month.
- *Baseball Cards*. I will go over the solution in next week's first lecture.

Week 30

- Deriving the *Change of Base Formula for Exponents*. Here are three different methods:
  In each case, we want to determine how 7^X can be converted into 3 raised to some exponent.
  In other words, how can $7^X = 3^Y$?

Method #1

$7^X = 3^Y$ take $\log_7$ of both sides

$\log_7 (7^X) = \log_7 (3^Y)$

$x = y \log_7 3$

$y = \frac{x}{\log_7 3}$ using $\frac{1}{\log_a b} = \log_b a$

$y = x \log_3 7$

$\therefore\ 7^X = 3^{x \log_3 7}$ and generally

$n^c = b^{c \cdot \log_b n}$

Method #2

Let $a = 7^X$

$\log_7 a = x$

$\therefore\ 7^X = 3^{x \log_3 7}$ and generally

$n^c = b^{c \cdot \log_b n}$

Since $7^X = 3^Y$, $a = 3^Y$

$\log_3 a = y$

$\log_3 (7^X) = y$

$x \log_3 7 = y$

using $7^X = 3^Y$ gives us

$7^X = 3^{x \log_3 7}$ and generally

$n^c = b^{c \cdot \log_b n}$

Method #3

Starting instead with

$n = b^{\log_b n}$

raising each side to c gives us:

$n^c = b^{c \cdot \log_b n}$

Week 31

Individual Work

1a) $3 + 3\sqrt{3}\, i$

b) $-½ + ½\, i$

c) $-5\, i$

d) $-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\, i$

e) $2 \cos(70°) + 2\, i \sin(70°)$
$\approx 0.684 + 1.879\, i$

f) $\approx -7.66 + 1.879\, i$

g) $\approx -6.88 + 2.45\, i$

h) $\approx 12.7 + 2.80\, i$

2a) $\approx 6.71 \text{ cis}(116.6°)$

b) $\approx 10 \text{ cis}(\pi/6)$

c) $\approx 7.62 \text{ cis}(203.2°)$

d) $\text{cis}(2\pi/3)$

e) $4 \text{ cis}(5\pi/4)$

f) $\sqrt{7} \text{ cis}(\pi/2)$

3a) $42 \text{ cis}(100°)$

b) $8 \text{ cis}(\pi) = -8$

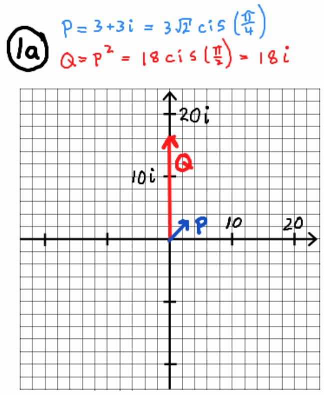
c) $50 \text{ cis}(31/12\ \pi) = 50 \text{ cis}(7/12\ \pi)$

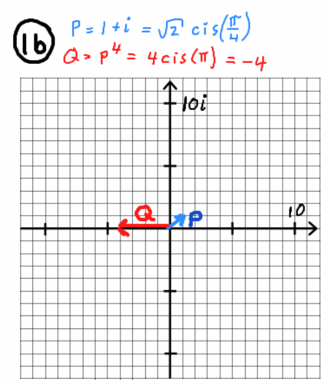
d) $16 \text{ cis}(200°)$

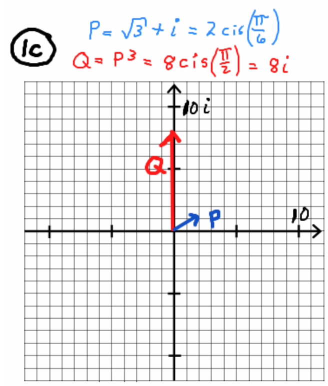
e) $-128 + 128\, i$

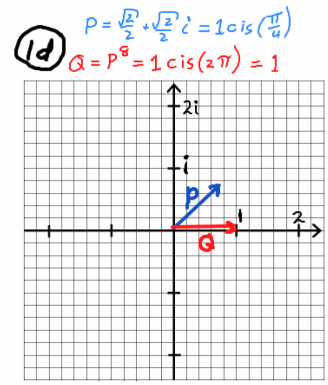
f) $16 + 16\sqrt{3}\, i$

(Week 31, cont.) Tuesday Group Work







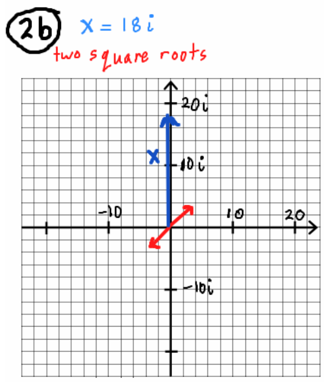


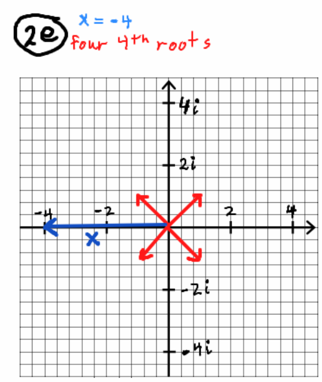
2a) 5, -5 b) **see below graph** $3\sqrt{2}$ cis($^{\pi}/_{4}$), $3\sqrt{2}$ cis($^{5\pi}/_{4}$) c) 3i, -3i d) ± 2, $\pm 2i$

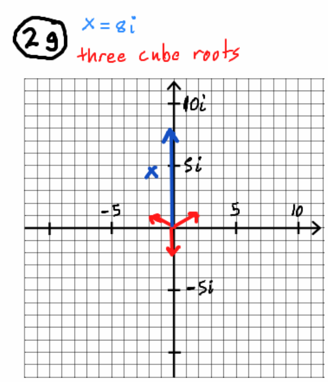
e) **see below graph** $\sqrt{2}$ cis($^{\pi}/_{4}$), $\sqrt{2}$ cis($^{3\pi}/_{4}$), $\sqrt{2}$ cis($^{5\pi}/_{4}$), $\sqrt{2}$ cis($^{7\pi}/_{4}$)

f) ± 1, $\pm i$, cis($^{\pi}/_{4}$), cis($^{3\pi}/_{4}$), cis($^{5\pi}/_{4}$), cis($^{7\pi}/_{4}$)

g) **see below graph** -2i, 2cis($^{\pi}/_{6}$), 2cis($^{5\pi}/_{6}$),







3 & 4) Follow these steps to find all of the roots: (1) take the root of the magnitude; (2) divide the angle by the degree of the root to get the angle of the first answer; (3) the rest of the answers have angles that are equally distributed.

5) In polar form: 1 cis (0), 1 cis ($^{2\pi}/_{3}$), 1 cis ($^{4\pi}/_{3}$). In rectangular form: 1, $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$, $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$

(Week 31, cont.) Thursday Group Work

1a) $3\,\text{cis}(\pi/4)$, $3\,\text{cis}(5\pi/4)$

b) $3\,\text{cis}(3\pi/4)$, $3\,\text{cis}(7\pi/4)$

c) $6\,\text{cis}(3\pi/8)$, $6\,\text{cis}(11\pi/8)$ <u>or</u> in rectangular form: $2.30 + 5.54\,i$, $-2.30 - 5.54\,i$,

d) $2\,\text{cis}(\pi/3)$, $2\,\text{cis}(5\pi/6)$, $2\,\text{cis}(4\pi/3)$, $2\,\text{cis}(11\pi/6)$

e) $5\,i$, $5\,\text{cis}(7\pi/6)$, $5\,\text{cis}(11\pi/6)$

f) $4\,\text{cis}(7\pi/12)$, $4\,\text{cis}(5\pi/4)$, $4\,\text{cis}(23\pi/12)$,

g) Polar Form — Rectangular Form

2.22	cis	7.44	°	2.20	+	0.29	*i*
2.22	cis	52.44	°	1.35	+	1.76	*i*
2.22	cis	97.44	°	-0.29	+	2.20	*i*
2.22	cis	142.44	°	-1.76	+	1.35	*i*
2.22	cis	187.44	°	-2.20	-	0.29	*i*
2.22	cis	232.44	°	-1.35	-	1.76	*i*
2.22	cis	277.44	°	0.29	-	2.20	*i*
2.22	cis	322.44	°	1.76	-	1.35	*i*

2) *The Hat-Check Problem.*

One approach is to start with just 1 person, create a table that builds up one person at a time, and then look for patterns. With the below table, P is the probability that no person gets their own hat.

For 1 person, we get $P = 0$.

For 2 people, we get $P = \frac{1}{2} = 0.5$

For 3 people, we get $P = \frac{2}{6} \approx 0.6667$

For 4 people, we get $P = \frac{9}{24} = 0.375$

For 5 people, we get $P = \frac{44}{120} \approx 0.3667$

For 6 people, we get $P = \frac{265}{720} \approx 0.36806$

For 7 people, we get $P = \frac{1854}{5040} \approx 0.36786$

For 8 people, we get $P = \frac{14833}{40320} \approx 0.36788$

We notice a few things. Firstly, the answer converges very quickly. In fact, quite surprisingly, once we get past 5 people, the probability (that nobody gets their own hat) remains unchanged to three significant digits. The denominators of P are n!, which might be expected because it is the number of ways of arranging n people. The pattern with the numerators is more tricky. If we call the n^{th} numerator h_n, then it turns out that:

$h_n = n \cdot h_{n-1} + (-1)^n$ and most interestingly, that $P_n = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} \ldots \pm \frac{1}{n!}$

Euler solved the problem and recognized that the solution (approximately 0.367879) approaches $1/e$, as the number of people (n) approaches infinity.

Week 32

Individual Work

1) $2\text{ cis}(30°)$, $2\text{ cis}(210°) \rightarrow$ which in rectangular form is $\sqrt{3} + i$ and $-\sqrt{3} - i$

2) $\sqrt[6]{200}\text{ cis}(45°)$, $\sqrt[6]{200}\text{ cis}(165°)$, $\sqrt[6]{200}\text{ cis}(285°)$
   which in rectangular form is $1.71 + 1.71\,i$, $-2.34 + 0.626\,i$, $0.626 - 2.34\,i$

3) $2\text{ cis}(0°, 60°, 120°, 180°, 240°, 300°) \rightarrow$ which in rectangular form is $\pm 2, \pm 1 \pm i$

4) $3\text{ cis}(^{\pi}/_{8})$, $3\text{ cis}(^{5\pi}/_{8})$, $3\text{ cis}(^{9\pi}/_{8})$, $3\text{ cis}(^{13\pi}/_{8})$
   which in rectangular form is $2.77 + 1.15\,i$, $-1.15 + 2.77\,i$, $-2.77 - 1.15\,i$, $1.15 - 2.77\,i$

5) $\sqrt[3]{5}\text{ cis}(53.13°)$, $\sqrt[3]{5}\text{ cis}(173.13°)$, $\sqrt[3]{5}\text{ cis}(293.13°)$
   which in rectangular form is $1.026 + 1.368\,i$, $-1.70 + 0.205\,i$, $0.672 - 1.57\,i$

Thursday Group Work

- *Three Shadows*.

This is a three-dimensional geometry problem, and a good preparation for matrix algebra and vector physics problems. All you really need to solve this problem is the Pythagorean Theorem.

The business about shadows is really a long way around saying the following: the rectangular block can describe a coordinate system, and that the length given on each face can be thought of as the result of the Pythagorean Theorem for the 2 dimensions on that face. If we imagine coordinates in x, y, and z at mutually orthogonal directions, then face A could be the x-z plane, B could be the x-y plane, and C the y-z plane. The Pythagorean Theorem states: $a^2 + b^2 = c^2$, where *a* is the change in coordinate length in one direction, *b* the change in length in the orthogonal direction, and *c* the total length in that plane.

Therefore, we can write:

Face A, equation 1	Δx^2		$+\Delta z^2$	$= 25$
Face B, equation 2	Δx^2	$+\Delta y^2$		$=36$
Face C, equation 3		Δy^2	$+\Delta z^2$	$=49$

One way to solve this is as follows:

$$\Delta x^2 + \Delta z^2 = 25$$
$$\Delta x^2 + \Delta y^2 = 36$$
$$\Delta y^2 + \Delta z^2 = 49$$

$$-\Delta y^2 + \Delta z^2 = -11 \text{ (subtracting eq.2 from 1)}$$
$$\Delta y^2 + \Delta z^2 = 49$$

$$2\Delta z^2 = 38 \text{ (adding the two above equations)}$$
$$\Delta z^2 = 19 \text{ result}$$

It then follows that: $\Delta x^2 = 6$; $\Delta y^2 = 30$; $\Delta z^2 = 19$

Therefore, the overall length of the rod is

$$l = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2} = \sqrt{6 + 30 + 19} = \sqrt{55}$$