# Answers for Grade 11 Group Assignments - Quarter #2

Notes:

- Answers for group assignment problems that are out of the workbook can be found in the "G11 – Workbook Answer Key".
- This answer key doesn't include all answers.

## Week 9 No answers needed.

## Week 10

- After some experimentation, we can list the factors of the perfect number in pairs. For example, using 28, 496 and 8128, we get:

| | | | | | |
|---|---|---|---|---|---|
| 1 | **28** | 1 | **496** | 1 | **8128** |
| 2 | 14 | 2 | 248 | 2 | 4064 |
| 4 | 7 | 4 | 124 | 4 | 2032 |
| | | 8 | 62 | 8 | 1016 |
| | | 16 | 31 | 16 | 508 |
| | | | | 32 | 254 |
| | | | | 64 | 127 |

Given that the number of rows is n, the last two numbers (which multiply to become the perfect number) are $2^{n-1}$ and $2^n - 1$, and so the perfect number, P, is given by $P = (2^{n-1})(2^n - 1)$. However, we must note that if n = 2, 3, 5, 7 then we get the perfect numbers P = 6, 28, 496, 8128, but if n = 4, 6, 8 and others, then this formula won't generate a perfect number. Why not? Well, in that case, we will get extra factors that aren't included in this pattern, and therefore the sum of the proper factors will be greater than the number itself, so the number will be *abundant* instead of *perfect*.

It then follows that the last number of the second column, namely $(2^n - 1)$, needs to be a prime number. Prime numbers of this form are known as Mersenne primes. So the question now becomes: what determines whether $(2^n - 1)$ is prime or not? Well, for one thing, n itself must be prime (we know this through one of the "laws" that we discovered on the previous problem). But other than that we need to check to see if it is prime. Of all of the prime values for n between 10 and 60, only n = 13, 17, 19, 31 produce values for $(2^n - 1)$ that are prime, which means…

**The 5$^{th}$ perfect number is: <u>33,550,336</u>**

The 6$^{th}$ perfect number is: 8,589,869,056

The 7$^{th}$ perfect number is: 137,438,691,328

The 8$^{th}$ perfect number is: 2,305,843,008,139,952,128

- ***Crazy Factoring! (Part I)***

$x^{12}-1 \rightarrow (x^6+1)(x^6-1) \rightarrow (x^6+1)(x^3+1)(x^3-1)$

Then we recognize perfect cubes to get:

$[(x^2+1)(x^4-x^2+1)][(x+1)(x^2-x+1)][(x-1)(x^2+x+1)]$

The binomial factors are: $(x+1)$, $(x-1)$, $(x^2+1)$, $(x^2-1)$,
$(x^3+1)$, $(x^3-1)$, $(x^4-1)$, $(x^6+1)$, $(x^6-1)$

Some possible laws are:

- If n is odd then $(x+1)$ is a factor of $(x^n + 1)$.
- For any n $(x-1)$ is a factor of $(x^n - 1)$.
- If i is a factor of n then $(x^i - 1)$ is a factor of $(x^n - 1)$.

## Week 11

1) **The Hiking Monk**

This problem is the same as the following scenario: Two monks start hiking at the same time – one from the bottom of the mountain going uphill, and the other from the top going downhill. What time do they meet?

As the original problem states, the trail is four miles long, they leave at 7am, and their speeds are in a 2:5 ratio. Looked at in this way, we can say that at the moment they meet, one monk has covered $^5/_7$ of the distance, and the other has covered $^2/_7$ of the distance. Therefore, they meet $^5/_7 \cdot 4$ $\approx$ 2.86 miles from the temple.

2) ***Crazy Factoring! (Part II)*** For $x^{15}-1$, we know that $(x-1)$, $(x^3-1)$, and $(x^5-1)$ must all be factors. We can start by using polynomial division to divide $(x^3-1)$ into $(x^{15}-1)$ to get: $x^{12} + x^9 + x^6 + x^3 + 1$, and then factoring the $(x^3-1)$, we finally get:

$$x^{15}-1 \rightarrow (x-1)(x^2+x+1)(x^{12} + x^9 + x^6 + x^3 + 1).$$

But why isn't $(x^5-1)$ in our final answer? To investigate this question, we start by dividing $(x^5-1)$ into $(x^{15}-1)$, or, better yet, we factor $(x^{15}-1)$ using perfect cubes to get:

$x^{15}-1 \rightarrow (x^5-1)(x^{10}+x^5+1)$, and then dividing $(x^5-1)$ by $(x-1)$, we get:

$$x^{15}-1 \rightarrow (x-1)(x^4+x^3+x^2+x+1)(x^{10}+x^5+1).$$

But above, we said that:

$$x^{15}-1 \rightarrow (x-1)(x^2+x+1)(x^{12} + x^9 + x^6 + x^3 + 1).$$

Why don't these two different approaches end up with the same result? It must be that neither has been completely factored. Of the four different polynomial factors, we can be confident that $(x^2+x+1)$ is prime, so it must divide evenly into (i.e., be a factor of) either $(x^4+x^3+x^2+x+1)$ or $(x^{10}+x^5+1)$. It ends up that $(x^2+x+1)$ doesn't go evenly into $(x^4+x^3+x^2+x+1)$, but it does divide into $(x^{10}+x^5+1)$, for an unexpected outcome of $(x^8-x^7+x^5-x^4+x^3-x+1)$. With some satisfaction we note that $(x^4+x^3+x^2+x+1)$ also divides evenly $(x^{12}+x^9+x^6+x^3+1)$ for a result of $(x^8-x^7+x^5-x^4+x^3-x+1)$.

We have finally completely factored $(x^{15}-1)$ to $(x-1)(x^2+x+1)(x^4+x^3+x^2+x+1)(x^8-x^7+x^5-x^4+x^3-x+1)$.

3) **Factoring $2^{60} - 1$**

Following some of the above ideas, we get:

$2^{60} - 1 = (2^{30} + 1)(2^{30} - 1)$ $(2^{30} + 1)(2^{15}+1)(2^{15}-1)$

$= [(2^{10}+1)(2^{20}-2^{10}+1)][(2^5+1)(2^{10}-2^5+1)][(2^5-1)(2^{10}+2^5+1)]$

This would be a good point to evaluate the parentheses:

[(1025)(1047553)][(33)(993)][(31)(1057)],

and then breaking these down further, we get:

$[(5^2 \cdot 41)(13 \cdot 61 \cdot 1321)][(3 \cdot 11)(3 \cdot 331)][(31)(7 \cdot 151)]$

Note that with a few of the larger numbers, we only need to try dividing the number by all of the prime numbers up to the square root of the number. With 1047553, we'd have to try dividing by all of the prime numbers up to $\sqrt{1047553}$, which is ≈1024. Dividing the class into groups, and giving them a table of prime numbers, quickly leads to the discovery that it's divisible by 13, leaving us with a quotient of 80581, and that number is divisible by 61, leaving a quotient of 1321. None of the primes up to $\sqrt{1321}$ divide evenly into 1321, so it must be prime.

**Our final answer is:** $3^2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 31 \cdot 41 \cdot 61 \cdot 151 \cdot 331 \cdot 1321$.

## Week 12

**Card Trick**

1) The key to this trick (assuming three piles and counting up to ten) is simply for the teacher to remember the seventh card that was seen when the original 26 cards were counted out. No matter what happens with the three piles, the final predicted card will always be this seventh card (which ends up being seven cards from the top of the original face-down discard stack).

2) Why is the "magic number" to find the predicted card always equal to seven? Here's an explanation. Let the values of the top cards in each of the three piles be $n_1$, $n_2$, $n_3$. The sum (T) of these three numbers is how far we count into the final discard stack to find the predicted card. The number of cards in each of the three piles is $(11 - n_1)$, $(11 - n_2)$, $(11 - n_3)$. Therefore, we can say that the total number of cards in the three piles is $33 - T$ (where $T = n_1 + n_2 + n_3$). Let X be the number of cards that have been added to the discard stack (after the initial 26 cards). We know that the total number of cards in the three piles $(33 - T)$ plus the number of cards that have been added to the discard stack (X) must be equal to 26. Therefore: $(33 - T) + X = 26$, which leads to $T - X = 7$. This tells us that the number of cards that we have to count into the final discard stack (T) (in order to find the predicted card) will always be seven greater than the number of cards that were added to the discard stack (X). In other words, the predicted card is always found at the seventh position down into the original (26-card) discard stack.

3) Let P be the number of piles, M be the number you count up to (when making the piles), and L be the number of cards left over after we count out the first D cards (so $D + L = 52$). Our goal is to determine C, the "magic number" that predicts the location of the desired card. In part b (above), we said that the total number of cards in the three piles was $33 - T$. Now it is $P(M+1) - T$.
We also said $33 - T + X = 26$, which is now $P(M+1) - T + X = L$. Therefore, the "magic number" $(C = T - X)$ we are looking for is

given by: $\mathbf{C = P(M+1) - L}$ The limitations are:

$C \le D$; $P(M+1) \le 52$; $P + D \le 52$; $P \le L < P(M+1)$

We can now perform the trick in this manner: Given a value for P, we can choose M such that $P(M+1) \le 52$, and then choose L.
For example, if we choose $P = 5$, then $M \le 9$. If we choose $M = 7$ (so when making the piles, any card above 7 counts as a 7), then $5 \le L < 40$. And if we choose $L = 30$, then $D = 22$ and **$\underline{C = 10}$**.
10 is the magic number for $D = 22$, $P = 5$, and $M = 7$.

## Week 13

1) a) $x^2 + 22x + 121 \rightarrow (x + 11)^2$

b) $x^2 + 2x + 1 \rightarrow (x + 1)^2$

c) $x^2 - 20x + 100 \rightarrow (x - 10)^2$

d) $x^2 + 7x + {}^{49}/_4 \rightarrow (x + {}^{7}/_2)^2$

2) a) $x^2 + 18x + 77 = 0$

$x^2 + 18x + 81 = -77 + 81$

$(x + 9)^2 = 4$

$x + 9 = \pm\sqrt{4}$

$x = -9 \pm 2$

$x = -11, -7$

b) $x^2 - 4x - 12 = 0$

$x^2 - 4x + 4 = 12 + 4$

$(x - 2)^2 = 16$

$x - 2 = \pm\sqrt{16}$

$x = 2 \pm 4$

$x = 6, -2$

2) c) $x^2 + 5x - 24 = 0$

$x^2 + 5x + \frac{25}{4} = 24 + \frac{25}{4}$

$(x + {}^{5}/_2)^2 = \frac{96}{4} + \frac{25}{4}$

$x + {}^{5}/_2 = \pm\sqrt{\frac{121}{4}}$

$x = -\frac{5}{2} \pm \frac{11}{2}$

$x = -8, 3$

3) a) $y = x^2 + 18x + 77$

$y = (x^2 + 18x + 81) + 77 - 81$

$y = (x + 9)^2 - 4$

vertex is (-9, -4)

b) $y = x^2 - 2x + 4$

$y = (x^2 - 2x + 1) + 4 - 1$

$y = (x - 1)^2 + 3$

vertex is (1, 3)

3) c) $y = x^2 + 7x - 3$

$y = (x^2 + 7x + \frac{49}{4}) - 3 - \frac{49}{4}$

$y = (x + \frac{7}{2})^2 - \frac{12}{4} - \frac{49}{4}$

$y = (x + \frac{7}{2})^2 - \frac{61}{4}$

vertex is (-3½, -15¼)

4) a) Center is (0,0); $r = 4$

b) Center is (-3,2); $r = 4$

c) $x^2 + 10x + y^2 - 8y + 32 = 0$

$(x^2 + 10x + 25) + (y^2 - 8y + 16) = -32 + 25 + 16$

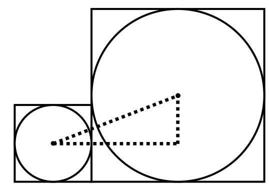
$(x + 5)^2 + (y - 4)^2 = 9$

Center is (-5, 4); $r = 3$

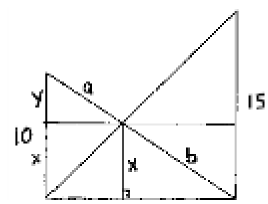
5) The graph of $\theta = ⅓$ is a line.
Graph of $r = 2$ is a circle.

## Week 14

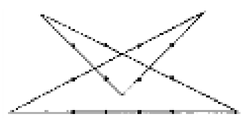
1) It is helpful to reframe the question, and instead ask ourselves, "Which circles *can't* be filled in with an X?" We can then see that only the bottom-left circle can't be assigned an X, for that would lead to having one row with three X's or three A's.
2) Perhaps the easiest way is to start by filling up the large bucket. Then you pour water from the large bucket into the small one, until the small bucket is full, and then discard the contents of the small bucket. Do this a total of three times, and you will be left with one gallon in the large bucket.
   Another possible solution is to pour water from the small bucket into the large bucket. Each time, the small bucket needs to be filled completely, and once the large bucket is full, the water should be discarded from the large bucket, which should leave two gallons in the small bucket. Pour that into the large bucket and then do it two more times, so you get 8 gallons in the large bucket. Then you simply fill the small bucket, and pour into the large bucket until it is full. There is now one gallon left in the small bucket.
3) The key is to construct a right triangle (as shown here) where the horizontal leg has a length equal to the sum of the circles' radii, and the vertical leg has a length equal to the difference of the circles' radii. The legs are therefore 24 and 10. By using the Pythagorean Theorem, or recognizing that these numbers are in a Pythagorean triple ratio (5-12-13), we can determine that the line connecting the circles' centers has a length of 26.



4) Draw a horizontal line through the top of the line marked originally as x. Using similar triangles we get a:b = 10:15 = 2:3, and from the *triangle proportionality theorem* we know that a:b = y:x. Therefore, y:x = 2:3. From this, we can say that x is $^3/_5$ of the whole line that has a length of 10. Thus, x = 6.




5) Here is the answer →→→→



## Week 15

1) Lacy, Stacy and Tracy may be 3, 4, 20; or 4, 6, 10; or 5, 8, 6; or 6, 10, 4. Maybe 2, 2, 60?

2) A) 15 B) 15 C) 40 D) 5050 E) 91 F) 18 G) 18 H) 3n I) b·n

3) A) This will be covered in the tutorial session.
B) $^3/_2$
D) n=6; x=4
Sum = 36,409

## Week 16

- Projective Geometry – No Answers Needed.