Answers
for Grade 12 Group Assignments - Quarter #1

Notes:

- Answers for group assignment problems that are out of the workbook can be found in the file named "G12 – Workbook Answers…".
- This answer key doesn't include all answers.

Week 1

- *Dog/Boat Puzzle.* (See separate file titled "G12 - W01 - Tutorial - Dog & Boat Puzzle")
- *Adding Digits* puzzle. Here are two (of perhaps several) possible approaches:

  Approach #1:
  0 through 9 adds to 45
  10 through 19 adds to 45 + 10·1
  20 through 29 adds to 45 + 10·2
  30 through 39 adds to 45 + 10·3
  etc., therefore we add the first ten groups of 10 to get
  0 through 99 adds to 10·45 + 10·(1+2+3…9) = 10·45 + 10·45 = 20·45 = **900**
  Similarly…
  0 through 99 adds to 20·45 + 100·0
  100 through 199 adds to 20·45 + 100·1
  200 through 299 adds to 20·45 + 100·2
  300 through 399 adds to 20·45 + 100·3
  etc., therefore…
  0 through 999 adds to 10·20·45 + 100·45 = 300·45 = **13500**
  Looking for patterns can be helpful:
  0 through 99 adds to 20·45
  0 through 999 adds to 300·45
  0 through 9999 adds to 4000·45
  0 through 99,999 adds to 50000·45
  So we can see the formula for adding all the numbers up to "n 9's":
  Total Sum = $n \cdot 10^{(n-1)} \cdot 45$
  Approach #2: (for adding all the digits of all the numbers up until 1,000,000)
  Before adding any digits, organize the numbers from 0 to 999,999 into pairs, as follows:
  0 999,999
  1 999,998
  2 999,997, etc. Here are a couple more pairs:
  38 999,961
  5107 994,892
  We can now see that the sum of the digits of each pair is 9·6 =54. There are 500,000 pairs, so the final sum is 500,000 · 54 = 27,000,000.
  And, therefore, the sum of the digits from 1 to 1 million is 27,000,001

Week 2

1c) $(x-5)^2 + (y+2)^2 = 13 \rightarrow C(5, -2);\ r = \sqrt{13} \approx 3.61$

1d) $y = 2x^2 - 20x + 43$ Using the method of "completing the square"

$y = 2(x^2 - 10x \quad) + 43 \rightarrow y = 2(x^2 - 10x \mathbf{+25}) + 43 \mathbf{- 50} \rightarrow y = 2(x-5)^2 - 7 \rightarrow V(5, -7);\ a=2$

1e) $y = -x^3 + \frac{1}{2}x^2 + 2x + 4$

To graph this, we can first ignore the 4, and then factor, to get

$y = -x^3 + \frac{1}{2}x^2 + 2x \rightarrow y = -\frac{1}{2}x(2x^2 - x - 4)$ which we can then put into the quadratic formula to get roots of $0, \frac{1 \pm \sqrt{33}}{4}$, or 0, 1.686, -1.186.

Now we can slide these points up by 4 (from the x-axis) in order to get these three key points on the original equation: (0, 4); (1.686, 4); (-1.186, 4)

Lastly, we can make a table to get a few more points, as desired.

2) Points of intersection are:

- **a** and **b** $\rightarrow$ **(-1.5, 5.5)**
- **a** and **c** $\rightarrow$ You can look at the graph, or use substitution. **(3,1); (8, -4)**
- **a** and **d** $\rightarrow$ $y + x = 4$ and $y = 2x^2 - 20x + 43$
  Using substitution gives us $x = 4 - y \rightarrow y = 2(4-y)^2 - 20(4-y) + 43$
  $y = 2y^2 - 16y + 32 - 80 + 20y + 43 \rightarrow 0 = 2y^2 + 3y - 5 \rightarrow 0 = (2y+5)(y-1) \rightarrow$
  y = 1, -2.5 and plugging these values into either of the original equations, gives us
  (3, 1); (-6.5, -2.5)
- **c** and **d** $\rightarrow$ $y = 2x^2 - 20x + 43$ and $(x-5)^2 + (y+2)^2 = 13$
  The trick here is that if you substitute the first into the second, you end up with a fourth degree equation. So instead, it is best to look for a better way to substitute.
  We notice that $y = 2x^2 - 20x + 43$ is also $y = 2(x-5)^2 - 7$,
  and here we see that $(x-5)^2$ is in both equations.
  We substitute the second equation [$\underline{\mathbf{(x-5)^2}} = 13 - (y+2)^2$] into the first $y = 2\underline{\mathbf{(x-5)^2}} - 7$,
  which gives us $y = 2[13 - (y+2)^2] - 7$
  This eventually simplifies to $2y^2 + 9y - 11 = 0$, which has solutions of y = 1, -5.5
  Plugging these values back into the original equations gives us
  points of intersection of (3, 1); (7, 1); (5.87, -5.5); (4.13, -5.5)
- **e** and **b** $\rightarrow$ $x + 3y = 15$ and $y = -x^3 + \frac{1}{2}x^2 + 2x + 4$
  The hint tells us that there is a point of intersection at (-1.5, 5.5)
  It may be best to solve the first equation for y, to get $y = -\frac{1}{3}x + 5$, and then substituting gives us: $-\frac{1}{3}x + 5 = -x^3 + \frac{1}{2}x^2 + 2x + 4$.
  We then get rid of the fractions by multiplying everything by 6 to get:
  $-2x + 30 = -6x^3 + 3x^2 + 12x + 24 \rightarrow 6x^3 - 3x^2 - 14x + 6 = 0$
  Factoring this is rather formidable, but we know that x = -1.5 is a solution, so (x + 1.5) must be a factor of the equation. (x + 1.5) produces the same solution as does (2x + 3).
  So now we can use polynomial long division to divide 2x+3 into $6x^3 - 3x^2 - 14x + 6$, which gives us $3x^2 - 6x + 2$. This means that $6x^3 - 3x^2 - 14x + 6 = (2x+3)(3x^2 - 6x + 2)$
  Lastly, we put $3x^2 - 6x + 2$ into the quadratic formula to get $x \approx 1.58, 0.423$
  The points of intersection are (-1.5, 5.5); (1.58, 4.47); (0.432, 4.86)

(Week 2, continued)

3) Number of Points of Intersection.

Each of the 10 lines has 9 points on it, but each point has two lines through it, so there are 9·10÷2 points of intersection. **Answer = 45 points of intersection**

Number of New Lines (that can be drawn)

Let point A be the point of intersection of lines x and y. There are 28 points (of intersection) that aren't on either lines x or y. Therefore, we can draw 28 new lines through point A. Since there are a total of 45 points of intersection, we can draw a total of 45·28÷2 new lines.
Answer = 630 new lines

4) First some background… If a number has the prime factorization $2^9 \cdot 3^8 \cdot 5^6 \cdot 13^4$, then we only need to look at the exponents of the 2 and the 5, in order to conclude that the number ends in 6 zeroes. Likewise, if a number's prime factorization is $2^7 \cdot 5^{13} \cdot 7^2 \cdot 11 \cdot 23^3$, then we know that that number must end in 7 zeroes.

Now, to address the question at hand… Let N = 4273! We know that the number of 2's in the prime factorization of N must be greater than the number of 5's (i.e., the exponent of the 2 must be greater than the exponent of the 5). Therefore, we simply need to determine the number of 5's in the prime factorization of 4273! We can systematically do this by asking ourselves the following questions:

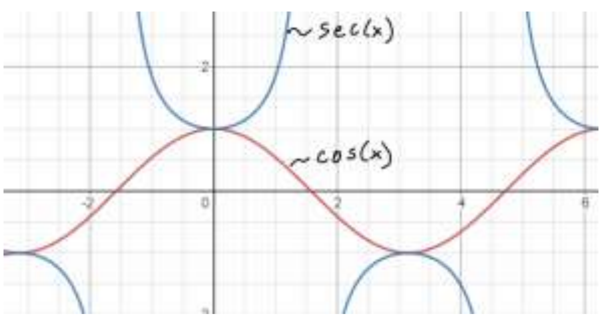
How many numbers between 1 and 4273 are…

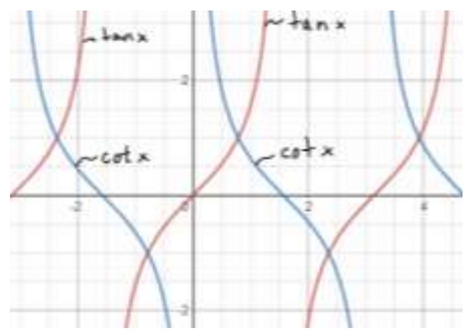
Divisible by 5?	Answer: 854.
Divisible by 25 (which is 5^2)?	Answer: 170.
Divisible by 125 (which is 5^3)?	Answer: 34.
Divisible by 625 (which is 5^4)?	Answer: 6.
Divisible by 3125 (which is 5^5)?	Answer: 1.

With a little bit of thought, we can now conclude that the number of zeroes in N must be equal to the sum of the above answers, which is 1065.

Week 3

For Tuesday:





Week 4

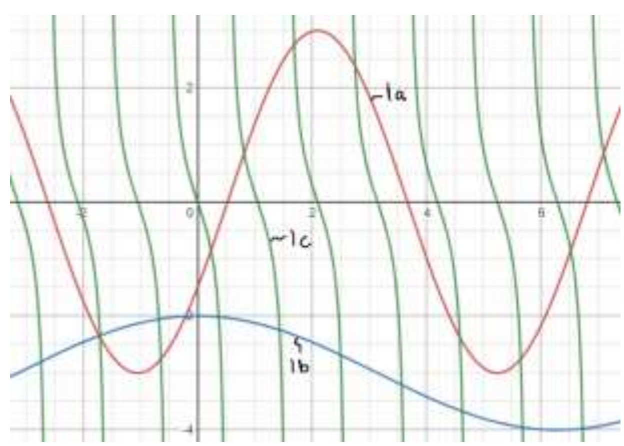
Four Sons. Let x_n be how much money the old man has after giving money to the n^{th} son.

$x_n = \frac{3}{4}(x_{n-1} - 4) = \frac{3}{4} \cdot x_{n-1} - 3$ which then gives us:

$x_1 = \frac{3}{4}x_0 - 3$; $x_2 = \frac{9}{16}x_0 - \frac{21}{4}$; $x_3 = \frac{27}{64}x_0 - \frac{111}{16}$

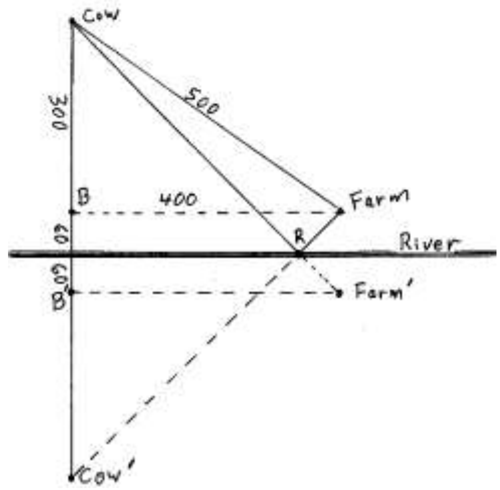
$x_4 = \frac{81}{256}x_0 - \frac{525}{64}$, which is also $x_4 = \frac{81x_0 - 2100}{256}$

where x_0 is the original number of coins, and x_4 is the number of coins after giving coins to the fourth son. So now the question is: what is the smallest positive integer that can be put into x_0 such that it will yield an answer for x_4 that is also a positive integer? Writing a simple computer program can show that the first integral answer to this Diophantine equation is **x_0 = <u>244</u> (and x_4 = 69)**.



Week 5

1. The Thirsty Cow.
   Not surprisingly, there are several ways to approach this problem. But, quite surprisingly, it turns out that reflecting the drawing across the river greatly reduces the level of complexity. We want to determine where R must be located such that the distance from Cow to R to Farm is minimized. It turns out that this is the same as asking what the shortest distance is from Cow to R to Farm', which clearly must be the straight line drawn from Cow to Farm'. We can then use the triangle Cow-Farm'-B' to calculate that the shortest distance (for the cow to get a drink and then get to the Farm) is 580m.



2. Tennis Tournament.
   For any single-elimination tournament, the number of matches must be one less than the number of participants. Therefore, for 100 participants, 99 matches must be played. This is because every match produces one loser, and every competitor except for one (the champion) walks away from the tournament with one loss.

Weeks 6-8

Answers to the *Calculus Discovery Sheets* are included in the main document.