JYMA: Solving a Cubic Equation

$$5x^3 - 30x^2 - 45x + 70 = 0$$ ← cubic

$$\frac{5}{5}x^3 - \frac{30}{5}x^2 - \frac{45}{5}x + \frac{70}{5} = 0$$

$$x^3 - 6x^2 - 9x + 14 = 0$$ ← monic cubic

let $x = y + 2$ ← 2 ← −⅓ · {coefficient of x^2}

$$(y+2)^3 - 6(y+2)^2 - 9(y+2) + 14 = 0$$

$$\begin{array}{cccc} y^3 & +6y^2 & +12y & +8 \\ & -6y^2 & -24y & -24 \\ & & -9y & -18 \\ & & & +14 \end{array} = 0$$

$$y^3 - 21y - 20 = 0$$ ← monic depressed cubic

let $y = v + 7v^{-1}$ ← 7 ← −⅓ · {coefficient of y}

$$(v+7v^{-1})^3 - 21(v+7v^{-1}) - 20 = 0$$

$$\begin{array}{ccccc} v^3 & +21v & +147v^{-1} & +343v^{-3} & \\ & -21v & -147v^{-1} & & \\ & & & & -20 \end{array} = 0$$

$$v^3 + 343v^{-3} - 20 = 0$$

$$v^3 \cdot [\, v^3 + 343v^{-3} - 20 = 0 \,]$$

$$v^6 - 20v^3 + 343 = 0$$

$$(v^3)^2 - 20v^3 + 343 = 0$$

let $v^3 = z$ (or… let $v = \sqrt[3]{z}$)

$$z^2 - 20z + 343 = 0$$ ← quadratic!

$$z^2 - 20z = -343$$

$$z^2 - 20z + 100 = -343 + 100$$

$$(z - 10)^2 = -243$$

$\sqrt{243}$ is $9\sqrt{3}$, but using $\sqrt{243}$ is easier. ↲

$$z = 10 \pm \sqrt{243}\, i$$

Rectangular form: $a + bi$

$z_{rectangular}$ → $a = 10,\ b = \sqrt{243}$

If a < 0, we are in quadrant II. In that case, we need to add π to θ to correct for the fact that $\tan^{-1}$ returns θ for quadrant IV. ↓ (Our a is positive; quadrant I is safe.)

Conversion: $r = \sqrt{a^2+b^2},\ \theta = \tan^{-1}\left(\frac{b}{a}\right)$

z_{polar} → $r = \sqrt{343},\ \theta = \tan^{-1}\left(\frac{\sqrt{243}}{10}\right)$ ← r is also $7\sqrt{7}$.

Polar form: $r \cdot \text{cis}(\theta)$

$$z = \sqrt{343} \cdot \text{cis}\left(\tan^{-1}\left(\frac{\sqrt{243}}{10}\right)\right)$$

$$v = \sqrt[3]{z}$$ ← recovering v

Polar roots: $\sqrt[n]{r \cdot \text{cis}(\theta)}$ → $\sqrt[n]{r} \cdot \text{cis}\left(\frac{1}{n} \cdot \theta\right)$

$$v = \sqrt[3]{\sqrt{343}} \cdot \text{cis}\left(\frac{1}{3} \cdot \tan^{-1}\left(\frac{\sqrt{243}}{10}\right)\right)$$ ← $\sqrt{343}$ is $\sqrt{7} \cdot \sqrt{7} \cdot \sqrt{7}$.

$$v_1 = \sqrt{7} \cdot \text{cis}\left(\frac{1}{3} \cdot \tan^{-1}\left(\frac{\sqrt{243}}{10}\right)\right)$$

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$$v_{1:polar} \rightarrow \; r = \sqrt{7},\;\; \theta = \frac{1}{3}\cdot\tan^{-1}\left(\frac{\sqrt{243}}{10}\right)$$

Conversion: $a = r\cdot\cos(\theta)$,

$b = r\cdot\sin(\theta)$

Rectangular form: $a + bi$

$$v_1 = \frac{5}{2} + \frac{\sqrt{3}}{2}i$$

$$v_1 = \frac{5+\sqrt{3}i}{2}$$

$$y = v + 7v^{-1} \qquad \leftarrow \text{recovering } y$$

$$y = v + \frac{7}{v}$$

$$y = \frac{v^2+7}{v}$$

$$y_1 = \frac{\left(\frac{5+\sqrt{3}i}{2}\right)^2 + 7}{\frac{5+\sqrt{3}i}{2}}$$

$$y_1 = \frac{\frac{25+10\sqrt{3}i-3}{4} + 7}{\frac{5+\sqrt{3}i}{2}}$$

$$y_1 = \frac{\frac{22+10\sqrt{3}i}{4} + \frac{28}{4}}{\frac{5+\sqrt{3}i}{2}}$$

$$y_1 = \frac{\frac{50+10\sqrt{3}i}{4}}{\frac{5+\sqrt{3}i}{2}}$$

$$y_1 = \frac{50+10\sqrt{3}i}{4}\cdot\frac{2}{5+\sqrt{3}i}$$

$$y_1 = \frac{10(5+\sqrt{3}i)}{4}\cdot\frac{2}{(5+\sqrt{3}i)}$$

$$y_1 = 5$$

$$x = y + 2 \qquad \leftarrow \text{recovering } x$$

$$\mathbf{x_1 = 7}$$

$$v_2 = \sqrt{7}\cdot \text{cis}\left(\frac{1}{3}\cdot \tan^{-1}\left(\frac{\sqrt{243}}{10}\right) + \frac{2}{3}\pi\right)$$

$$v_2 = -2 + \sqrt{3}i$$

$\vdots$

$$\mathbf{x_2 = -2}$$

$$v_3 = \sqrt{7}\cdot \text{cis}\left(\frac{1}{3}\cdot \tan^{-1}\left(\frac{\sqrt{243}}{10}\right) + \frac{4}{3}\pi\right)$$

$$v_3 = -\frac{1}{2} - \frac{3\sqrt{3}}{2}i$$

$\vdots$

$$\mathbf{x_3 = 1}$$

<u>Notes</u>:

- Once we have determined v_1, we can find v_2 and v_3 by multiplying by a complex cube-root of 1.

  That is…

  $$v_2 = v_1\cdot\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$

  …and…

  $$v_3 = v_2\cdot\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$

- Once we have found x_1, we can divide our monic cubic by $(x - x_1)$ to reduce it to a quadratic, which can make finding x_2 and x_3 easier.

  For example…

  $$\frac{x^3 - 6x^2 - 9x + 14}{x - 7}$$

  …via long division becomes…

  $$x^2 + x - 2$$

- If we suspect a polynomial has a rational root, we can employ the *rational root theorem.* Notice how our roots (7, −2, 1) are factors of the constant term (14) of our monic cubic? That's not a coincidence! ☺