12^{th} Grade Assignment – Week #31

GROUP WORK – TUESDAY / After L1 and Before L2:

[I] *Linear Iteration – Seeds, Orbits, and their Fates*

Here is an example linear iteration rule: $x \to 4x + 2$. It tells us that for any given x, its next value will be equal to $4x + 2$. We can then take that result and iterate repeatedly.

Our initial choice, x_0, is called the *seed*. The sequence of values we reach from iterating (x_0, x_1, x_2, x_3, …) is called its *orbit*. The behavior of the orbit as we approach x_∞ is called its *fate*.

To write a formula on a calculator that makes use of the previous result, we can take advantage of the [Ans] key. If we wish to iterate $4x + 2$ with seed $x_0 = 5$, we begin by entering 5, so that it will be the result, and then write our formula with [Ans]. We can then recall that entry to iterate.

```
5
                5
4×Ans+2
               22
4×Ans+2
               90
```

There can be real value in working like this. However, if at any point you want to speed up the process, **here is a link to a Google Sheet** that will.

For each of the following rules, determine the fate of the orbits for the given seeds.

❶ Rule: $x \to \frac{1}{2}x + 2$	
Seed	Fate of orbit
$x_0 = 0$	
$x_0 = 6$	
$x_0 = 8$	
$x_0 = -10$	
$x_0 = -4$	
$x_0 = 4$	

❷ Rule: $x \to 2x - 2$	
Seed	Fate of orbit
$x_0 = 0$	
$x_0 = 1$	
$x_0 = 3$	
$x_0 = -5$	
$x_0 = 5$	
$x_0 = 2$	

❸ Rule: $x \to -2x + 3$	
Seed	Fate of orbit
$x_0 = 0$	
$x_0 = -1$	
$x_0 = 10$	
$x_0 = -6$	
$x_0 = -10$	
$x_0 = -\frac{1}{4}$	

❹ Rule: $x \to -\frac{1}{2}x + 3$	
Seed	Fate of orbit
$x_0 = 0$	
$x_0 = -4$	
$x_0 = 10$	
$x_0 = 4$	
$x_0 = 100$	
$x_0 = -46.3$	

12^{th} Grade Assignment – Week #31 (continued)

<u>GROUP WORK – TUESDAY</u> <u>/ After L1 and Before L2</u>: • *Linear Iteration (continued)…*

[II] *Linear Iteration – Fixed Points*

Choose your own seed values to explore the remaining rules. Can you find any *fixed points*?

A *fixed point* is anywhere an orbit gets stuck. If the value you put into the rule is the value you get out, you've found a fixed point.

You don't have to guess. You can find these algebraically. Replace "→" with "=" and solve.

❺ Rule: $x \to -x + 1$	
<u>Seed</u>	<u>Fate of orbit</u>
$x_0 =$	
$x_0 =$	
$x_0 =$	
$x_0 =$	

❻ Rule: $x \to 1.5x + 1$	
<u>Seed</u>	<u>Fate of orbit</u>
$x_0 =$	
$x_0 =$	
$x_0 =$	
$x_0 =$	

❼ Rule: $x \to -0.6x$	
<u>Seed</u>	<u>Fate of orbit</u>
$x_0 =$	
$x_0 =$	
$x_0 =$	
$x_0 =$	

Fixed points can be *attracting* (they draw seeds toward themselves), *repelling* (values move further away at each iteration), or *neutral* (neither attracting nor repelling).

Do all linear iteration rules have a fixed point? Did you, or can you, find fixed points for all seven of the rules we have explored? Is it possible for a linear iteration rule to have no fixed points, or to have multiple? And if so, under what conditions would that happen?

[III] *Linear Iteration – Parameters of Possibility*

Rather than only investigating linear iteration rules individually, we would like to understand the behavior of this entire class. That is, given any linear iteration rule whatsoever, we would like to be able to draw conclusions about it based solely on its parameters A and B.

$$x \to Ax + B$$

You will have noticed patterns in the rules we have worked with already, and may already have ideas about how A and B influence the eventual outcome of a rule. We don't need to answer this question completely right now, and working through the next section will help clarify aspects of it. However, if you have conjectures at this stage, you might want to take a moment to test them. Create your own rules and see if they behave as you anticipate. Play around, but not for too long.

12th Grade Assignment – Week #31 (continued)

<u>GROUP WORK – TUESDAY</u> <u>/ After L1 and Before L2</u>: • *Linear Iteration (continued)…*

[IV] *Linear Iteration – Time Series*

While orbits can be represented as sequences (x_0, x_1, x_2, x_3, …), graphing can be a powerful way to visualize their behavior. A graph which plots the values for an orbit in sequence is called a *time series* graph. Each tick-mark on the horizontal axis corresponds to an increment in the subscript for x, our iteration counter. The vertical axis shows the value for x at each iteration. Although sometimes the points are connected with lines to emphasize the sequencing, there are not any actual values between them. There is no such thing as a fraction of an iteration.

Time series graphs are shown for seven different linear iteration rules (① – ⑦).

The orbit of two different seeds (• & ∗) are given for each.

Each rule is of the form $x \rightarrow Ax + B$.

Match each of the time series graphs to the A value (Ⓐ – Ⓖ) that belongs to it.

Ⓐ $A = -1$

Ⓑ $0 < A < 1$

Ⓒ $A < -1$

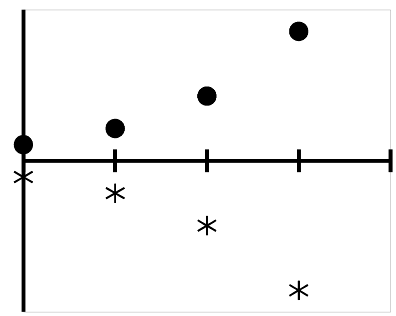
Ⓓ $-1 < A < 0$

Ⓔ $A > 1$

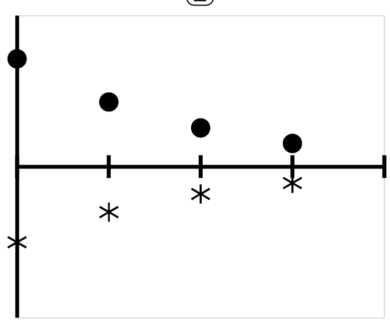
Ⓕ $A = 1$

Ⓖ $A = 0$

①



②



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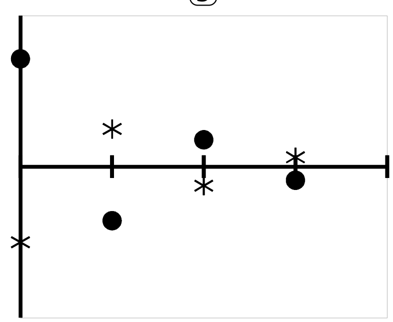

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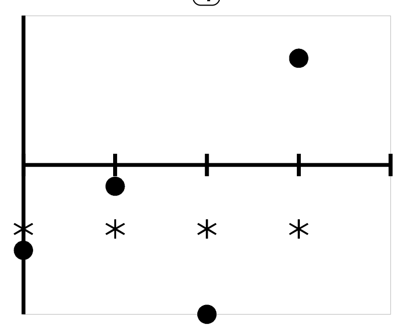
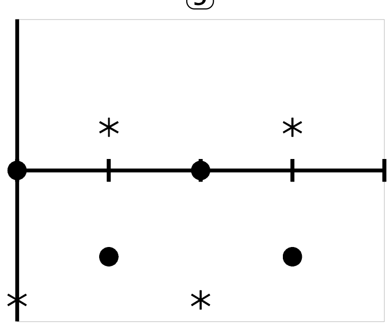
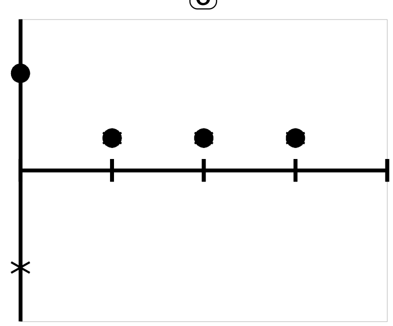


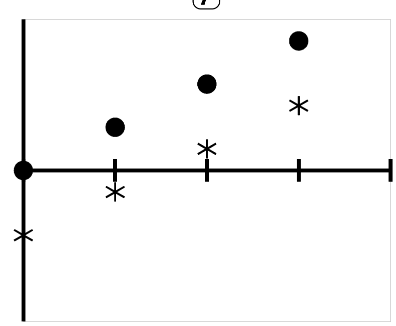
⑤



⑥



⑦



[V] *Linear Iteration – Parameter Plane*

Return to our question from [III] *Parameters of Possibility.* Can you completely describe the behavior of a linear iteration rule based on its parameters A and B?

Consider making a map of your conclusions. Sketch a plane with A as the horizontal axis and B as the vertical axis, then divide it into regions of specific kinds of behavior.

12^{th} Grade Assignment – Week #31 (continued)

GROUP OR INDIVIDUAL WORK – JUST FOR FUN

St. Petersburg Game

At the end of Lecture 1, two games of chance were briefly discussed. One was a relatively ordinary coin toss, and the other, the St. Petersburg game, was decidedly more curious.

On the next page is a summary of those two games and what I asked.

1. ORDINARY GAME
   A coin is tossed. If it comes up HEADS, you win \$200; if it comes up TAILS, you win nothing.

   What can we expect the average winnings for playing this game to be? There are only two possible outcomes, each with equal probability: P(HEADS) = ½ …and… P(TAILS) = ½.

   Thus we can say the expected average winnings = ½(\$200) + ½(\$0) = \$100.

   If you only play the game once, of course, you either win \$200 or win nothing. But if you play the game over and over again forever, your average winnings per game approaches \$100.

   What would you be willing to pay to play this game? We can use the calculation above as a guide. Any wager less than \$100 we can expect, over time, to result in positive winnings.

2. ST. PETERSBURG GAME
   In this game, you toss a coin repeated until it comes up HEADS, after which you stop. Every game earns you something, and how much you win is given by how many TAILS you tossed.

   HEADS → \$2

   TAILS, HEADS → \$4

   TAILS, TAILS, HEADS → \$8

   TAILS, TAILS, TAILS, HEADS → \$16

   TAILS, TAILS, TAILS, TAILS, HEADS → \$32

   TAILS, TAILS, TAILS, TAILS, TAILS, HEADS → \$64

   TAILS, TAILS, TAILS, TAILS, TAILS, TAILS, HEADS → \$128

   …and so on, with every game continuing until the coin comes up HEADS.

   What can we expect the average winnings for playing this game to be?
   What would you be willing to pay to play this game?

GROUP WORK – THURSDAY / After L2:

Non-Linear Iteration – Logistic Function

Here is the logistic function. It is quadratic (and therefore non-linear). We are interested in how the behavior of iterating this function, the fate of orbits, for changes in the parameter k.

$$x \to kx\cdot(1-x)$$

Here is a link to a Google Sheet to aid your investigations. For any k and x_0 you enter, it gives data through x_{1023}, a time series graph, a histogram, and the function itself atop an identity line.

12^{th} Grade Assignment – Week #31 (continued)

GROUP WORK – THURSDAY / After L2: • *Logistic Function (continued)…*

[VI] *Logistic Function – Preliminaries*

You may skip directly to the next section *(Playing with k)* if you please. This section serves as a reference which you may or may not choose to return to as you proceed through your analyses.

x-Range: $0 \leq x \leq 1$

For the context we are considering, values outside this range are meaningless. Given that, here is what algebra can tell us. If you were inspired, you could prove all this later, but don't do so now.

k-Range: $0 \leq k \leq 4$

The parabola's vertex (where $x = ½$) yields its peak value. We can set this value to the 1, the maximum possible, and solve for k. This informs us as to the acceptable range for parameter k.

Fixed Points: $x = \frac{k-1}{k}$, and $x = 0$. (Remember, only valid for $0 \leq x \leq 1$.)

If we let $x = kx \cdot (1-x)$, we can solve for the fixed points given above.

2-Cycle Points: $x = \frac{k^2 + k \mp k\sqrt{(k+1)(k-3)}}{2k^2}$. (Imaginary solutions are invalid in our context.)

Solving for the above is a challenge. Plugging the function into itself gives a quartic (4^{th}-power) equation. Fortunately, by factoring out the initial fixed points, we can reduce this to quadratic. It is not far from here, however, that we reach the end of what algebra gives us the power to solve.

[VII] *Logistic Function – Playing with k*

For the following, use seed $x_0 = 0.5$. Just watch out for the value of k for which this is fixed.

Parameter k	Fate of orbit for $x_0 = 0.5$
$k = 0.2$	
$k = 0.4$	
$k = 0.8$	
$k = 1.1$	
$k = 1.3$	
$k = 1.8$	
$k = 2.0$	

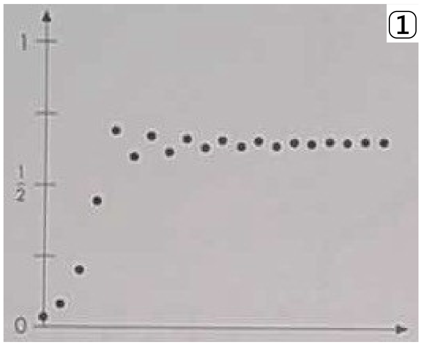
Parameter k	Fate of orbit for $x_0 = 0.5$
$k = 2.5$	
$k = 3.1$	
$k = 3.2$	
$k = 3.4$	
$k = 3.5$	
$k = 3.554$	
$k = 3.7$	

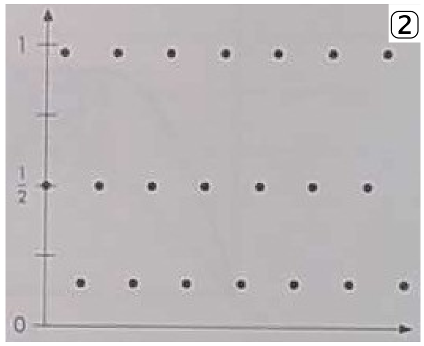
12th Grade Assignment – Week #31 (continued)

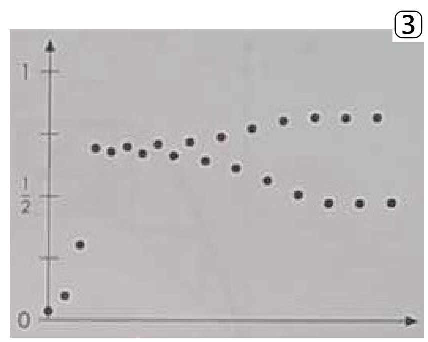
GROUP WORK – THURSDAY / After L2: • *Logistic Function (continued)…*

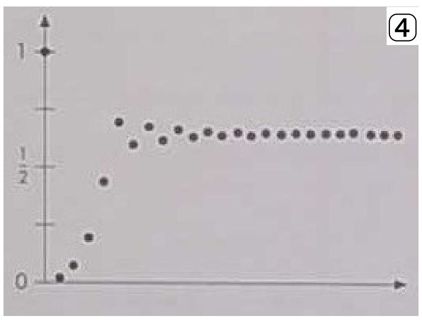
[VIII] *Logistic Function – Time Series & Graphical Iteration*

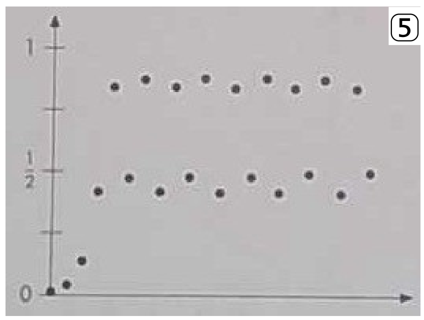
Match each of the follow time series with the appropriate graphical iteration picture.

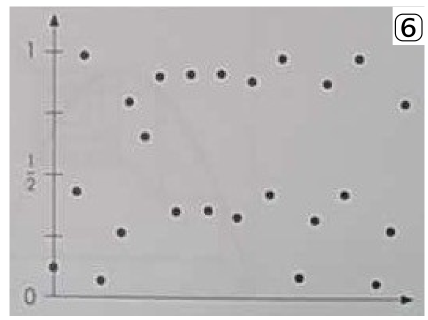


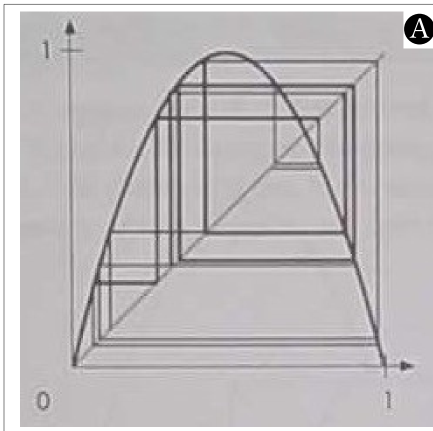


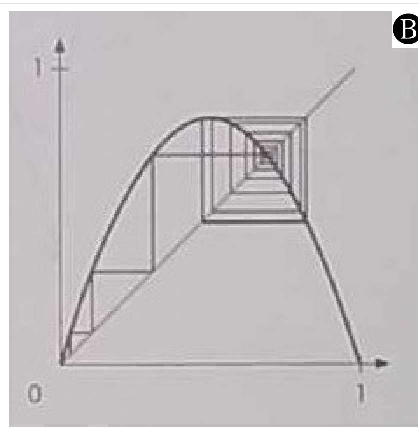


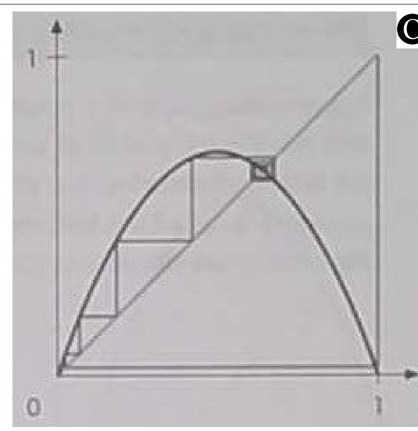


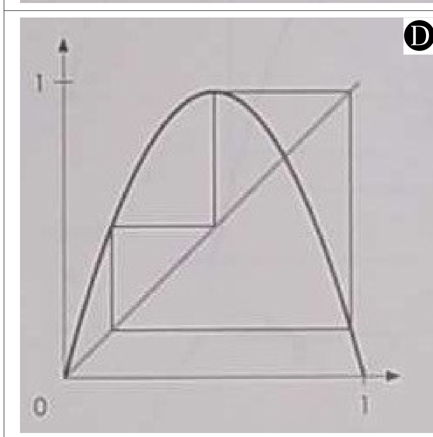


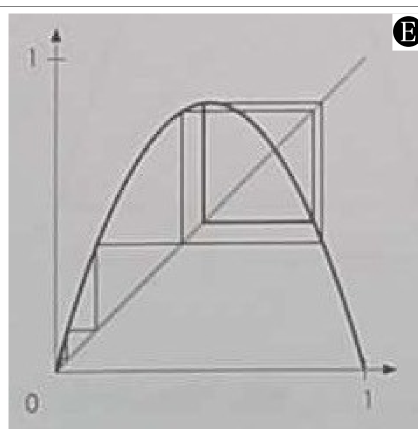


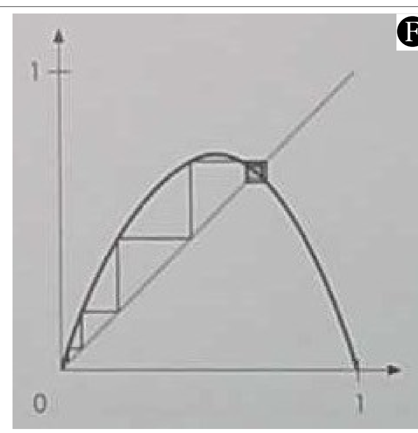












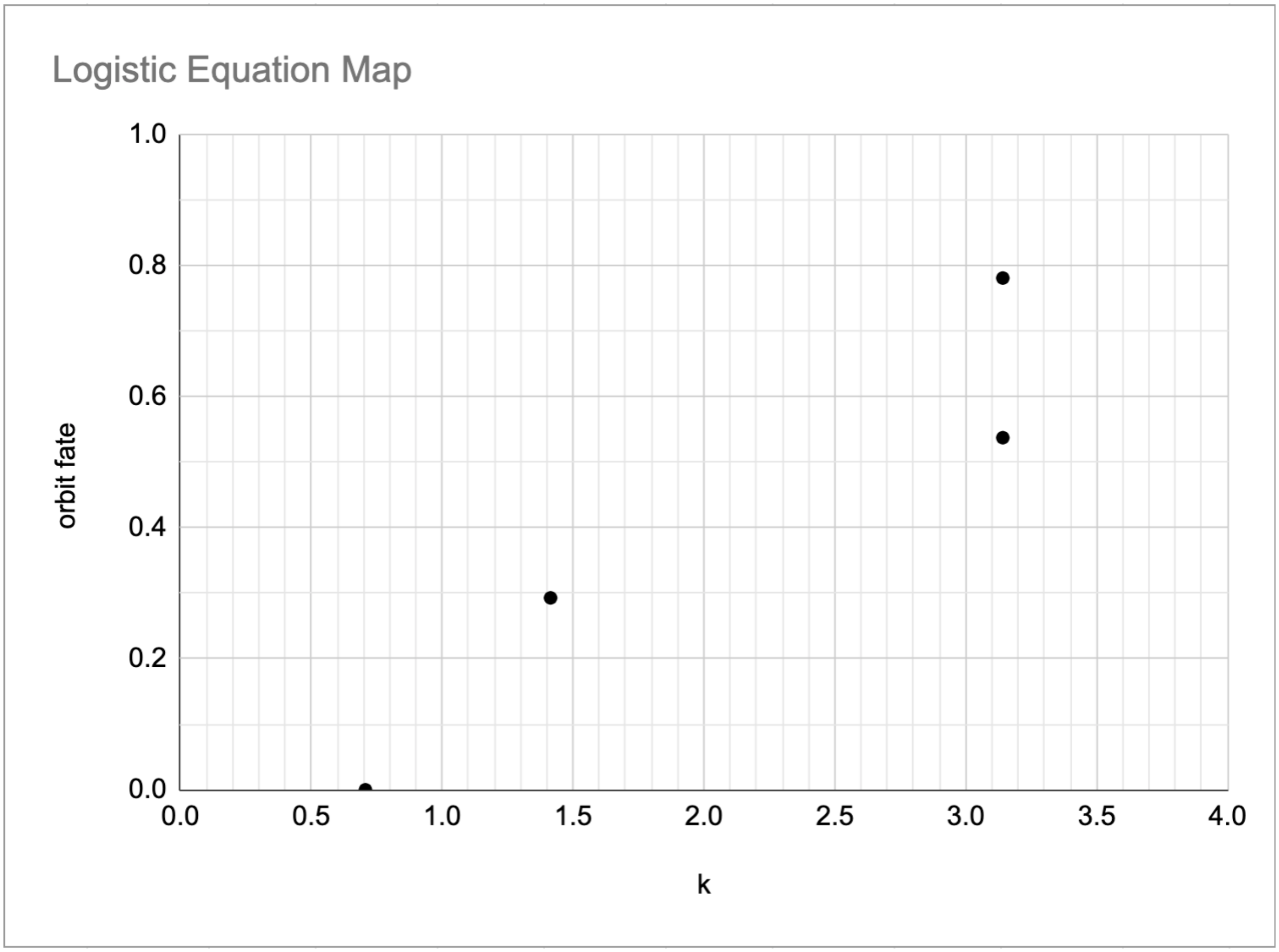
12th Grade Assignment – Week #31 (continued)

<u>GROUP WORK – THURSDAY</u> <u>/ After L2</u>: • *Logistic Function (continued)…*

[IX] *Logistic Function – Making a Map*

Here is the beginning of a map describing the behavior logistic function. (It does not include the transient behavior with which many orbits begin, but attempts to show the fate of such orbits.)

The values we found at the end of the last lecture are plotted. Adding the results of your work in the previous section will improve the picture. You're welcome to explore further too. We'd like to understand the logistic function's behavior as completely as we understand linear iteration.



If you'd like a full-page version of this to work with, **<u>here is a link to a graph you can print</u>**.

[X] *Logistic Function – One Last Peek*

In next week's lecture, courtesy of the computational effort of others, we will look at a far more detailed map than you could plot by hand in this way. Before you end your investigation, here is one last set of values to enter into the spreadsheet. I consider the results quite interesting.

$k = 3.84$	$x_0 = 0.1167$	Fate of orbit?

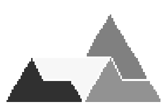
12^{th} Grade Assignment – Week #31 (continued)

Individual work

- **Watch this 7½-minute excerpt from *Arcadia*.**
- Readings: Keep up with the published list of readings.
- Cubic work: Going from x → y → v → z. Given the following cubic equation…

$$2x^3 + 6x^2 - 24x - 10 = 0$$

  - Divide through by the coefficient of x^3 to make it monic (coefficient of $x^3 = 1$).
  - Then… let $x = y - \frac{1}{3} \cdot \{\text{coefficient of } x^2\}$ …to depress it (no y^2 term).
  - Then… let $y = v - \frac{1}{3} \cdot \{\text{coefficient of } y\} \cdot v^{-1}$, thus employing the Vieta substitution.
  - You now have v^3 and v^{-3} terms. Multiply the equation by v^3 to make all exponents positive.
  - Finally… let $v^3 = z$ …to end up with a quadratic equation in terms of z.
  - You can check your work with the “CF” tab in any recent iteration of our spreadsheet.
- Continue any group work you would like to take further.
- Work on main lesson pages / artistic creations of your own design as suits.
- And if you’re inclined, download and play with our **Complex Plane Fractals web page**. ☺

Selected Answers

[IV] ①🅔, ②🅑, ③🅓, ④🅒, ⑤🅐, ⑥🅖, ⑦🅕 —or— 🅐⑤, 🅑②, 🅒④, 🅓③, 🅔①, 🅕⑦, 🅖⑥

[VII] k = 0.2 → $x_\infty = 0$ k = 0.4 → $x_\infty = 0$ k = 0.8 → $x_\infty = 0$ k = 1.1 → $x_\infty = 0.\overline{09}$

k = 1.3 → $x_\infty \approx 0.2308$ k = 1.8 → $x_\infty = 0.\overline{4}$ k = 2.0 → $x_\infty = 0.5$ k = 2.5 → $x_\infty = 0.6$

k = 3.1 → $x_\infty \approx 0.5580, 0.7646$ k = 3.2 → $x_\infty \approx 0.5130, 0.7795$ k = 3.4 → $x_\infty \approx 0.4520, 0.8422$

k = 3.5 → $x_\infty \approx 0.3828, 0.8268, 0.5009, 0.8750$ [4-cycle]

k = 3.554 → $x_\infty \approx 0.3521, 0.8108, 0.5453, 0.8812, 0.3720, 0.8303, 0.5007, 0.8885$ [8-cycle]

k = 3.7 → $x_\infty \approx$ 0.????

[VIII] ①🅕, ②🅓, ③🅑, ④🅒, ⑤🅔, ⑥🅐 —or— 🅐⑥, 🅑③, 🅒④, 🅓②, 🅔⑤, 🅕①

[X] k = 3.84 → $x_\infty \approx 0.1494, 0.4880, 0.9594$ [3-cycle]