12th Grade Assignment – Week #30

GROUP WORK – TUESDAY / After L1 and Before L2:

[I] *Chaos Game*

At the end of Lecture 1, we took a very quick look at the Chaos Game. (That really is the name which is used for it. If you'd like to "play" with it yourself, **here is a link to our Chaos Game Gallery**. It is an HTML file which, after downloading, you can open in a web browser to run.)

0. YOUR BEST GUESS
   While each version of the game is its own question, your assignment is to focus on the two games played with half jumps and no restrictions. When there were three vertices to choose from, the Sierpiński Triangle appeared. But when there were four vertices to choose from, no fractal appeared; eventually the entire square would be filled in. Why do you think this is so? Share your thoughts with the group.

1. SIERPIŃSKI TRIANGLE
   Here are the instructions for this version of the game.

   To start…
   - Plot the vertices of an equilateral triangle, and name each vertex (perhaps A, B, and C).
   - Let your starting point be any point at all. Put a dot there.

   Repeat the following forever…
   - Randomly choose a vertex (perhaps a die roll of 1 or 2 for A, 3 or 4 for B, and 5 or 6 for C).
   - Your next point is halfway between your current point and the chosen vertex. Plot it.
   - If this isn't at least your twelfth turn, erase the previous dot.

   Somewhat astonishingly, given all the randomness involved, this version of the game will always create the Sierpiński Triangle. To help you analyze the game, **here is a link to an 8th-iteration Sierpiński Triangle**. Print it out and consider its corners you vertices A, B, and C.

   A. Rather than choosing randomly, start in the exact center of the Sierpiński Triangle you printed out. Now again, rather than choosing a random vertex to move toward, let's move toward each. Create three new points, one halfway from the center to each of A, B, and C. Then, from each of those three points, plot all possible moves ($3 \times 3 = 9$ new points).

      Make a hypothesis about what is going on, and then test it by following points further. What can we say about the fate (that's a mathematical term too), the infinite endpoint of a point that begins in the exact center and follows the rules?

   B. Now let us analyze the fate of a different point. Let your starting point now be the upper vertex. As before, plot all the possible paths it could take for a couple iterations, and then selectively follow some further if that helps you picture what is occurring. Develop and test a conjecture. What can we say about the end result of starting at the top vertex?

   C. If possible, generalize your conclusions from above to say whatever can be said about starting from anywhere at all. Note that our second trial began with a point on the fractal perimeter, whereas the starting point for our first trial was not.

   D. Why does this Chaos Game generate the Sierpiński Triangle? And why, in the plotting of many tens of thousands of points, would it be meaningful to discard the first dozen?

12^{th} Grade Assignment – Week #30 (continued)

<u>**GROUP WORK – TUESDAY** / After L1 and Before L2</u>: • *Chaos Game (continued)…*

2. NON-FRACTAL SQUARE

This was the first game we played. Here are the instructions for this version of the game.

To start…

- Plot the vertices of a square, and name each vertex (perhaps A, B, C, and D).
- Let your starting point be any point at all. Put a dot there.

Repeat the following forever…

- Randomly choose a vertex (perhaps using a tetrahedral die or tossing a coin twice).
- Your next point is halfway between your current point and the chosen vertex. Plot it.
- If this isn't at least your twelfth turn, erase the previous dot.

This game does not produce a fractal. The entire square is slowly filled in.

As above, rather than choosing randomly, choose a meaningful starting point and follow each of its paths forward. Try to determine what is going on, demonstrate the truth of it, and justify your reasoning to the group.

[II] *Complex Calculator, Part 2*

Our previous work has prepared us for this next step in, shall we say, complexity. Our goal is to program the spreadsheet to evaluate this expression here. ⟶ $\dfrac{2x^3 + 1}{3x^2}$

As creating such a formula in a single spreadsheet cell would be unwieldy, and we have ample cells to spare, we will approach this in steps. Here is a screenshot of the evaluation.

	A	B	C	D	E	F	G
1	Initial Input Line	:	INPUT	4.000	+	-2.000	i
2							
3			x = ↑	4.000	+	-2.000	i
4			x^2	12.000	+	-16.000	i
5			x^3	16.000	+	-88.000	i
6			2x^3	32.000	+	-176.000	i
7			2x^3 + 1	33.000	+	-176.000	i
8			3x^2	36.000	+	-48.000	i
9	Final Result Line	:	(2x^3 + 1) / (3x^2)	**2.677**	+	**-1.320**	i
10							

[Here is a link to a Google Sheet](#) to get you started. The "N3" sheet is where you will find this. You may wish to refer to our previous complex calculator work in sheet "CC". *When creating your formulas, do not refer to cells D1 and F1 directly.* Instead, refer to cells D3 and F3 when you need to use those values. (It doesn't matter now, but will in the next iteration of this work.)

12^{th} Grade Assignment – Week #30 (continued)

GROUP OR INDIVIDUAL WORK – Just For Fun

Something Odd About Pascal's Triangle

This is a playful curiosity, and an opportunity to color while engaging in a free conversation regarding all we have studied thus far. That said, given the above tasks, you may not have any time for this. That's okay. Going through this exercise will be a topic for the week's tutorial.

Here is a link to a Pascal's Triangle template. Actually, it's just a bunch of circles, but in two different sizes. Whether you go for the big or small circles depends on how far you want to take things. Pascal's Triangle, of course, just like most math teachers, can go on forever. Regardless of the size you choose, use the page in landscape orientation, and start with a bubble near the center of the top row. Oh, and in case you've forgotten, Pascal's Triangle begins this way…

```
        1
      1   1
    1   2   1
  1   3   3   1
1   4   6   4   1
```

…with each new number summing the two which are above it (empty space is equal to zero).

Now, you could try to squeeze ever bigger numbers into the bubbles, but that is not the point of this exercise. Instead, all we are interested in is whether a given number is odd or even, which can be done without needing to keep track of the sums.

Give the bubble for any odd number a dark color, and the bubble for any even number a lighter color. That's it. Whether you fill in the bubbles neatly with an aesthetic color scheme or just put a quick dark pencil scribble in each odd one is up to you. Whatever you do, enjoy the process.

GROUP WORK – THURSDAY / After L2:

[III] *Complex Calculator, Part 3*

Newton's Method is an iterative approach to finding the roots of any function. It begins with a guess, and then uses this formula → to determine the next guess, iterating the process as needed.

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

$f(x) = x^3 - 1$ ← The roots of this function are the cube roots of 1. We can use Newton's Method to find those roots. Using the formula with our function as well as

$f'(x) = 3x^2$ ← its derivative yields the following specific rule. →

$$x_n = \frac{2x_{n-1}^3 + 1}{3x_{n-1}^2}$$

If you ignore the confusing subscripts, it may look familiar. It is exactly what we have built our spreadsheet to calculate. Progressing from the one evaluation we have now to repeated iteration only requires copying the block of formula cells. This is why, rather than linking all our formulas to the input row, we created a new row for "x" to refer to. That row always looks two rows up for its values, so if you leave an empty row between blocks, when you paste a new block in its starting values will be the ending values from before.

Here is a link to a Google Sheet to get you started. It is what you created with the addition of an iteration counter. As you copy-and-paste the code block, the counter will increase. Once you have multiple blocks pasted, you can copy them all and thus paste multiple blocks. (I chose to go up to 32 iterations.) When you're done, split the pane such that the input row is always visible.

12^{th} Grade Assignment – Week #30 (continued)

<u>Group work – Thursday</u> <u>/ After L2</u>: • *Complex Calculator, Part 3 (continued)…*

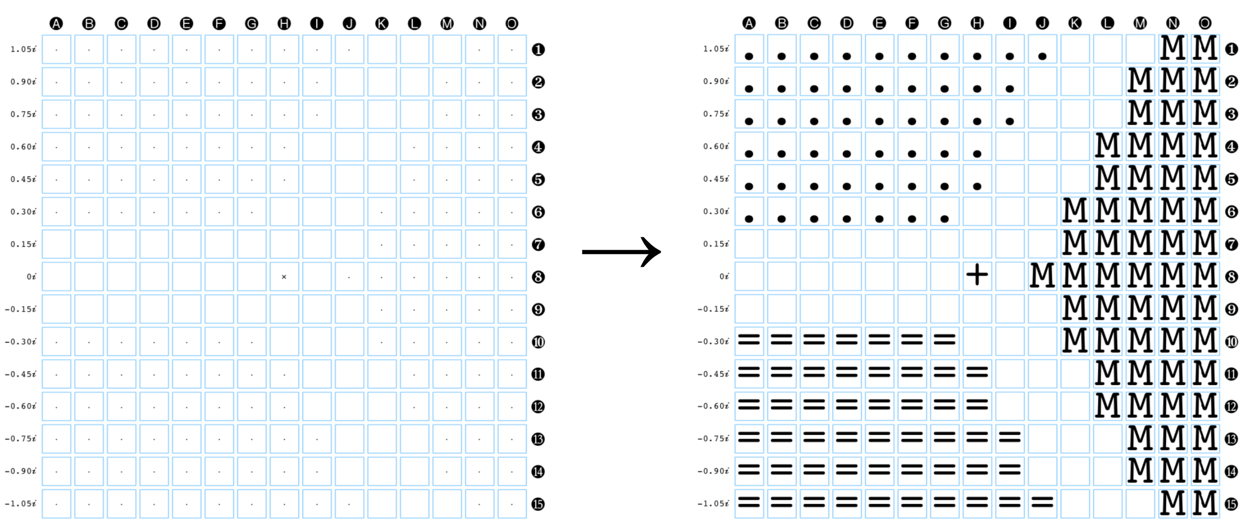
Below you can see that, if we start with $\mathbf{x_0 = 4 - 2}\boldsymbol{i}$, at the seventh iteration (x_7) we have found that $\sqrt[3]{1} = 1$. Astounding! Well, okay, maybe not, but remember, our goal here is to discover which specific cube root of 1 is reached by Newton's Method for any given starting value.

	A	B	C	D	E	F	G
1	Iteration	0	INPUT	4.000	+	-2.000	i
50							
51		7	x = ↑	1.002	+	0.000	i
52		7	x^2	1.004	+	0.000	i
53		7	x^3	1.007	+	0.000	i
54		7	2x^3	2.013	+	0.000	i
55		7	2x^3 + 1	3.013	+	0.000	i
56		7	3x^2	3.013	+	0.000	i
57	Iteration	7	(2x^3 + 1) / (3x^2)	**1.000**	**+**	**0.000**	**i**
58							

[IV] *Newton's Method and The Cube Roots of One*

If you did not complete or lack confidence in the above, **here is a link to a Google Sheet** that's all ready to go. We will use our formula on a 15×15 grid of points in the complex plane, creating a map of which root each of those points find. Our grid goes from $\mathbf{-1.05 - 1.05}\boldsymbol{i}$ (lower left) to $\mathbf{1.05 + 1.05}\boldsymbol{i}$ (upper right). **Here is a link to the map you are to complete**. Please print it out.

We know that starting near any root will lead us to that root. There are little dots in the cells for all those points which have been calculated for you. You are going to focus on what is between. Before we do so, though, it would be nice to code the values we already know. To be certain it would print well for everyone, I used characters. Coloring the cells would create a nicer picture.



12^{th} Grade Assignment – Week #30 (continued)

<u>GROUP WORK – THURSDAY /</u> <u>After L2</u>: • *Newton's Method and The Cube Roots of One (continued)…*

Now we can begin. Choose an unknown cell in the grid and read its value from the axes. Enter it into your spreadsheet, let the computer iterate Newton's Method for you, and see what it finds.

Here are the cube roots of one, and how they appear in our spreadsheet. Mind the tiny minus!

1	$\longrightarrow$	**1.000**	**+**	**0.000**	**i**
$-\frac{1}{2}+\frac{\sqrt{3}}{2}i$	$\longrightarrow$	**-0.500**	**+**	**0.866**	**i**
$-\frac{1}{2}-\frac{\sqrt{3}}{2}i$	$\longrightarrow$	**-0.500**	**+**	**-0.866**	**i**

For convenience while collaborating, note that **0.45 + 0.75*i*** can be referred to as map cell **K3**.

<u>INDIVIDUAL WORK</u>

- **<u>Watch this 2½-minute excerpt from *Arcadia*</u>**.
- <u>Make up a cubic equation depress it</u>. (You can check your work against the "CF" tab in any recent iteration of our spreadsheet. The tab also shows the expression to substitute in for x.) It would be good to feel comfortable with this process before the next step comes in tutorial.
- <u>Continue any group work you would like to take further</u>. (The first lecture next week (#31) will show why the *Sierpiński Tetrahedron (Tetrik)* fractal deserves its dimensionality, explain why the *Chaos Game* produces what it does, and reveal the complete map for our *Newton's Method and The Cube Roots of One* work.)
- <u>Work on main lesson pages as suits</u>. (While I find depictions of successive iterations of the Sierpiński Arrowhead Curve to be lovely, you might not. As you know, there is no grade for this main lesson. Without dismissing the value of representing ideas concretely, and how we grow through that work, choose what will be most meaningful for you. We have more content to come, but do think about how you might capture some essence of our studies artistically, be that visually, verbally (a reflective personal essay, a poem, a three-hour play), or otherwise.)
- <u>Plan to read</u>. Your first <u>readings</u> are to be completed before next week's second lecture. ☺