12th Grade Assignment – Week #28

Announcement

- Next week, you will begin our main lesson: *Fractal Geometry &Chaos*. Please do not look at the Week #29 assignment until after watching the first lecture of Week #29. In preparation for that lecture, please **download and print this template**, and have the following available:
  - • Compass • Ruler • Colored Pencils • Ordinary Pencil • Eraser • Blank Paper
  - • Pen (optional) • 30°-60°-90° Triangle (optional)

Individual Work

- Read Chapter #6 of *Logicomix* after you have watched this week's first lecture (which concludes Gödel's proof), but before Tuesday's group meeting.

Group Assignments:

for Tuesday.

- Questions regarding Gödel's proof. I want you to discuss your opinion on each question. I'll go over the questions in the next lecture.
  1. What was the "quest"? How was this affected by Gödel's proof?
  2. What are the implications and consequences of Gödel's proof?
  3. What did Gödel think of his proof?
  4. What did the Vienna Circle think?
  5. What did the greater world of mathematics think of the proof?
  6. How did Gödel's proof affect Hilbert's goal?
- Discussion for *Logicomix* reading, chapter #6
  7. Even though Russell was often in favor of change, what was his fear regarding modern art (brought about by "angry artists")?
  8. What smells of "metaphysical bosh", and of "sour grapes"?
  9. What did Gödel say (to Russell) was the most basic assumption – something which Russell had not made clear in PM?
  10. What was Wittgenstein teaching the little children? (Look on the blackboard of his classroom!)
  11. What was Russell's new approach to teaching children?
  12. Fill in these statements (from Gödel's talk at the logical conference in Vienna):
"There will always be ______."
"Arithmetic, and thus any system based on it, is, of necessity, _______."
  13. According to Russell, both Nazism and Communism are extremes, and have what in common?
Do you think that the political far left and far right have this (or anything else) in common today?
  14. At the end, what was Russell's advice to the audience about whether to join the war, or not?

for Thursday.

- Questions about general philosophy.
  - What are the central questions of (general) philosophy?
  - Is philosophy important? Is it important how a society views the world? Why?
  - Give examples of when a dominant philosophy (of the world/life) influences our daily lives and politics?
- Questions about this course.
  - Do you think it was good to learn about the philosophy of math? Why?
  - Do you think it was good to learn about Gödel's proof? Why?

Gödel's Proof Day #3 – The Central Argument

Note: This proof uses formulas B and G (see previous sheet, *Gödel's Proof – Functions & Formulas*).
What follows is only an outline of the central argument of the proof.[1]

The following steps are justified by using formulas, axioms, and rules from within PM.

1. Assumption: PM is consistent, which means Formula B is true.
2. G says, "G is not provable." Its negation, ~G, says, "G is provable".
   - (For a moment, ignoring what G says) if, somehow, G can be proven, then G is provable, which is what ~G says, so it has also been proved. In summary, *if G is provable, then ~G is provable*.
   - If, somehow, ~G can be proven, then because ~G says "G is provable", G must also be provable. In summary, *if ~G is provable, then G is provable*.

   (Gödel proved the two above statements within the system of PM.)
3. We just said that *if* G is provable, then ~G is provable. We also know (from Key Idea #3, previous page) that "*If a formula and its negation can both be proven as true, the system is inconsistent.*" Then it follows that if G is provable, then the system (PM) is inconsistent. This allows us to conclude (using Key Idea #7, previous page): If PM is consistent, then *G is not provable*.

Now, using meta-mathematical reasoning…

4. We have just shown that G is not provable (assuming that PM is consistent). But, G says "G is not provable." So now we know that *G is true*. Therefore, *G is true, but not provable within PM*.
5. We have now found a statement, expressible within PM, that is true but not provable. We also know that: "*With any mathematical system, if there exists one true formula that is not provable within the system, then that system is incomplete.*" (Key Idea #3, previous page)

Conclusion #1: *If PM is consistent, then it is incomplete.*[2, 3]

6. The statement "PM is incomplete" is the equivalent of saying: "There exists one true formula of PM that is not provable." We have now discovered that G is such a formula. Therefore, we can now say that G has the following new meaning: "*PM is incomplete*".
7. Assumption: Formula B (which says: "PM is consistent") is provable.
8. We now can say the following:
   - From Conclusion #1: "If PM is consistent, then PM is incomplete"
   - Which also means: "If B [is true], then G [is true]."
   - Which leads us to: "If B is provable, then G is provable."
9. In summary, if we assume that PM is consistent (B is true, step #1) and assume that the consistency is provable (B is provable, step #7), then it must follow (from step #8) that *G is provable*.
10. However, we said in step #3 that if PM is consistent, then G is *not* provable. We have a contradiction, which means that our assumption (step #7) is incorrect.
    Therefore B (which says: "PM is consistent") *is not provable*.

Conclusion #2: *If PM is consistent, then the consistency cannot be proven.*[4]

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[1] For more details, read *Gödel's Proof* (by Ernest Nagel and James Newman, NYU Press, 2001, 2nd edition) for an excellent account of Gödel's Proof.

[2] Gödel also proves that PM is *essentially* incomplete, which means that even if the system is "repaired" by adding more axioms so that it can handle any problematic formulas (like G), then another true, but unprovable formula can always be constructed. And Gödel proves that this is true of *any* formal, axiomatic system that encompasses the elementary properties of whole numbers, including addition and multiplication.

[3] The meta-mathematical statement, "If PM is consistent then PM is incomplete" (step #5), can be expressed within PM by the formula (which can be abbreviated $B \supset G$):

$$[(\exists w) \sim(\exists z)\, Dem(z,w)] \supset [\sim(\exists x)\, Dem(x,[sub(n,17,n)])]$$

[4] To be more precise, I quote Ernst and Nagel (p107): "If PM is consistent, its consistency cannot be proven by any meta-mathematical reasoning that can be mirrored within PM itself. It does *not* exclude [the possibility of] a meta-mathematical proof of the consistency of PM. What it excludes is a proof of consistency that can be mirrored inside of PM…Proofs that cannot be mirrored inside the systems that they concern are not finitistic; they [therefore] do not achieve the proclaimed objectives of Hilbert's original program."

Notes from the Last Lecture of *The Philosophy of Math* (the consequences of Gödel's proof)

- I tell younger students that the last thing I teach in 12th grade is the proof that "ended math as we knew it". Did it end math? What did it really mean to the world of mathematics?
- **"The quest is impossible!!!"** Gödel's proof ended the quest to find the perfect mathematical foundation.
- What are the implications of the proof? What did he actually prove? (See previous page for conclusions.)
  - Gödel also proved that PM is *essentially* incomplete, which means that even if the system is "repaired" by adding more axioms so that it can handle any problematic formulas (like G), then another true, but unprovable formula can always be constructed.
    <u>This destroyed the dream of Russell and PM</u>.
  - And Gödel proves that this is true of *any* formal, axiomatic system that encompasses arithmetic.
    <u>This destroyed the objectives of Hilbert's Program.</u>
- Math, as we knew it, has changed. It demands that we think of math differently.
- Hilbert (and Russell) wanted to give a firm foundation for all mathematics by eliminating intuition from mathematics. Gödel showed that math cannot proceed without intuition.
- Does Gödel's proof say that we can't prove anything to be true?
  - No, we can still prove things to be true.
  - His proof says that we can't have a perfect axiomatic system, nor a perfect foundation for mathematics.
  - "Truth" has not changed. But it seems to be that intuition has to play a part in truth.
- The sad irony is that even though Gödel had accomplished what his dream was, he was misunderstood. The logical positivists saw it as evidence of the meaninglessness of math.
- Goldstein's book, *Incompleteness*:
  - "Some thinkers despaired at this result. Others could never accept it. And still others misunderstood it as a torpedo to the hull of rationality itself. For Gödel, however, it was evidence of an eternal, objective truth, independent of human thought, that can only be imperfectly apprehended by the human mind."
  - "It is extraordinary that a mathematical result can say anything about the nature of mathematical truth in general (meta-mathematics)."
  - "[Gödel believed] that mathematical reality must exceed all formal attempts to contain it."
  - The formalists and positivists were saying that math is a meaningless formal game, reduced to "manipulating meaningless symbols" according to the rules of the game. This game could be played by a machine. Gödel's theorems seem to imply that our minds are not just machines.
- This proof doesn't mean that math is any less true than before Gödel. It just means that our traditional means for proving things (i.e., axiomatic systems) are not as flawless as we thought.
- Gödel's proof doesn't say that we can't prove anything (e.g., 3+5 = 8). Of course, arithmetic <u>is</u> true. It does say that we can't prove that an axiomatic system (which includes arithmetic) is consistent.
- "We know that God exists because mathematics is consistent and we know that the devil exists because we cannot prove the consistency." -- Andre Weil
- As humanity was cutoff more and more from the spiritual world, math education became less philosophical & more mechanical.
- The Logical Positivist's (and Logistic School's) central theme that "mathematics is meaningless" is still, in many ways prevalent today, and has had a big impact on math education.

- <u>From my HS Sourcebook</u>: Many people misunderstood the implications of Gödel's proof. Gödel "defeated" Russell and Hilbert, but sadly, the logical positivists (Vienna Circle) saw it as further evidence of the meaninglessness of math, and this attitude continues to have an impact on mathematics education today.

After Sputnik (the Soviets' first satellite in 1957), America became focused on beating the Soviets. Through this narrow lens, the purpose of mathematics education became to produce more scientists in an effort to win this mad race. Over the past few decades, the world has changed in many ways – socially, economically, and politically. Perhaps now more than ever, there is a realization that mathematics education needs to be overhauled. Opinions differ about where to place the blame, and how to fix it. New mathematics curricula have come and gone. Math textbooks are thicker, "prettier" (more graphics and fancier "packaging"), and more expensive – but less effective than ever. The trends are clear: an increase in teaching to standardized tests; more reliance on technology-driven learning; and teaching more advanced material at a younger and younger age.

And the results? A superficial understanding of material; a loss of a "sense of number"; an inability to do simple calculations in your head; no time left for depth, discovery, or true problem solving. In short, mathematics has become meaningless for most students.

But we must not give up. We all need to become math missionaries. We need to embark on a crusade to make math meaningful.