12^{th} Grade Assignment – Week #27

Announcement

- The next three lectures focus on Gödel's Proof. I will include notes for these lectures in this document, below, and in next week's assignment, as well.

Individual Work

- Read Chapter #5 of *Logicomix* before Tuesday's group meeting.
- Read Barry Mazur's paper (*Mathematical Platonism and its Opposites*) before Thursday's group meeting.

Group Assignments:

for Tuesday.

- Answer these questions about Gödel numbers.
  See the "Notes for Gödel's Proof – Day #1, **Table of Symbols**", on the next page.

  1) What is the formula represented by the below the Gödel number:
     2,890,087,473,357,680,023,474,838,649,646,985,115,279,360,000,000
     Hints: Use an online "large number calculator", such as this.
     13 is the largest base in the prime factorization.
     The number of zeroes tells you the smaller exponent between the bases 2 and 5.

  2) Consider the formula
     $(x=s0) \supset (\exists y)(y=ssss0)$
     a) What is the Gödel number for this formula (in prime factorization form)?
     b) Translate the formula into an English statement.

- *Logicomix* Discussion (Chapter #5)
  - How would you describe Wittgenstein?
  - (p237) What is meant by "Europe was sick with nationalism"? What's with the dominoes?
  - (p241-2) What did Wittgenstein learn about language by looking at the military map?
  - Why did Beetle join the army? Why did Wittgenstein join?
  - (p257-261) When BR was reunited with Wittgenstein, what did he think of Wittgenstein's new book (Tractatus) and his new thoughts about philosophy?

for Thursday.

- Discussion on Mazur's paper: "*Mathematical Platonism and its Opposites*"
  - p34: What's the problem with the stealth word, "our"?
  - p34: What are the two main camps regarding "the question"? Led by whom?
  - p35: Is the Platonic view really a theistic stance?
  - p35: For the Platonist, what is the role of proof in mathematics?
  - Do you think that the author is a Platonist?

- Answer this question*:* What do you think Gödel will end up proving?

- **Puzzle:** Give a proof (or an explanation) for this statement: "*For any prime number greater than 3, one less than its square must be divisible by 24.*" For what other numbers will this work?

Notes for Gödel's Proof – Day #1

Table of Symbols within PM and *<u>Gödel numbering</u>*

Symbol	Gödel #	Meaning	Symbol	Gödel #	Meaning
~	1	not	(	8	left parenthesis
$\vee$	2	or	)	9	right parenthesis
$\supset$	3	if…then…	,	10	comma
$\exists$	4	there exists…	+	11	plus
=	5	equals	$\cdot$	12	multiplication
0	6	zero	x	13	variable
s	7	successor of	y	17	variable
			z	19	variable

- **<u>Note</u>:** ss0 stands for 2.
- **<u>Examples</u>** of proper formulas include:
  - x = ssss0, which means "x equals 4"
  - sssss0 = ss0 + ss0, which means "5 equals 2 + 2"
  - ($\exists$x)(x = ss0 + sss0), which means "there exists an x such that x equals 2+3"
  - ($\exists$x)(x = ss0) $\supset$ ($\exists$y)(~(y = 0)), which means "If there exists x such that x equals 2, then there exists y such that y is not equal to 0."

Gödel's Proof Day #2 – Key Ideas, Functions, and Formulas

I. Key Ideas

0. *Terms*. PM, Consistent, Complete, Converse, Contrapositive.
1. *Truth and Certainty*. Whether a statement is true or not is a different matter than whether or not the statement can be proven as true or false. For example, the statement "Lions are mammals" may be true, but it cannot be proven within PM.
2. *Meta-Mathematics*. Meta-mathematics is when we "stand above" a mathematical system and make statements about it. Examples of meta-mathematical statements include: "Euclid proved 465 theorems in *The Elements*"; "PM is consistent." Quite amazingly, Gödel found a way to create formulas within PM that were, at the same time, meta-mathematical statements about PM.
3. *If a formula and its negation (opposite) can both be proven as true, the system is inconsistent.*
4. *If a system is inconsistent, then any formula (even one that is obviously false) can be proven as true*. For example, *if* it could be proven within a system that $2+2=4$ and that $2+2\neq 4$, *then* there is a contradiction, and therefore the system is inconsistent. It would then follow that any formula within the system could be proven as true (e.g., $4=5$; $7=8$; $21=24$, etc.).
5. *If, within a given mathematical system, there exists a (presumably false) formula that cannot be proven as true, then the system is consistent.* This is the contrapositive of the above statement.
6. *In order to prove that a system is consistent, we only need to find one (presumably false) formula within the system that cannot possibly be proven as true.* For example, if we can prove, by using the axioms of the system, that there is no possible way to prove that $3+9=7$, then we have proven that the system is consistent. (Note that because the consistency of PM is under question, simply proving that $3+9 = 7$ is false is not sufficient to be able to conclude that it can't be proven true.)
7. *Given the statement: "If E is true then F must be true", if we happen to know that F is false, then we can conclude that E cannot be true.*
8. *With any mathematical system, if there exists one true formula that is not provable within the system, then that system is incomplete.*

II. Functions. Gödel found a way to express both of the following functions within PM.

Dem(z,w) "Dem" is short for "demonstrate", which is another word for "prove". It answers the question: "Is z a proof of w?" This function is true only if the sequence of formulas given by the Gödel number z is a proof of the formula given by the Gödel number w. (Formula w would be the last formula in proof z.)

Sub(a,b,c) "Sub" is short for "substitute". This is a bizarre but important function. a is the Gödel number of a formula. b is the Gödel number of a symbol. c simply represents a symbol. Sub(a,b,c) tells us to take the formula with Gödel number a, find all occurrences of the symbol indicated by Gödel number b, and then replace them with the symbol c. The function Sub(a,b,c) returns the Gödel number of the resulting formula.

Example: If the formula x = y + ss0 is given by the Gödel number k, then sub(k,13,0) returns the Gödel number of the formula 0 = y + ss0.

III. Formulas. The key to Gödel's brilliance is that he finds a way to express certain important meta-mathematical statements about PM within the system of PM. He does this by using two amazing statements – each one is a meta-mathematical statement that is expressed within PM.

(Note that for now, we are not saying whether either formula is true, or not.)

Formula B: (∃w) ~(∃z) Dem(z,w)

This reads as: "There exists a formula w such that there does *not* exist a proof (z) of w", or: "w is not provable". Using Key Idea #5, above, Formula B is therefore the equivalent of the meta-mathematical statement: **"PM is consistent."**

Formula G: ~(∃x) Dem(x,[sub(n,17,n)])

This statement is very important, but difficult to comprehend. You must first know that n is the Gödel number of the formula ~(∃x) Dem(x,[sub(y,17,y)]). Formula G then reads, "There does *not* exist an x such that x is a proof of the formula given by the Gödel number that results from sub(n,17,n)." In other words, the statement that results from sub(n,17,n), *which is Formula G itself*, is not provable. Formula G is therefore the equivalent of the meta-mathematical statement: **"Formula G is not provable."** (Note that at this point we don't know whether G is true or not.)

The two above formulas are used as centerpieces in the central argument for Gödel's proof, which you will see in the next lecture.