11^{th} Grade Assignment – Week #32

Individual Work

- Do the following problems:
  1) Find the two square roots of $2 + 2\sqrt{3}\,i$.
  2) Find the three cube roots of $-10 + 10\,i$.
  3) Find the six sixth roots of 64.
  4) Find the four fourth roots of $81\,i$.
  5) Find the three cube roots of $3 + 4\,i$
- Finish anything from the "Group Assignment" that your group doesn't complete.

Group Assignment:

for Tuesday

- Together, work through Problems #3-6 on **Problem Set #6.**

for Thursday (Choose from any of the below.)

- **Finishing Complex Numbers.** If you didn't finish Problem Set #6 on Tuesday, then you can do so now.
- **The Cubic Formula**
  - Together, read through *Formulas for Solving Polynomial Equations* (see next page). Notes:
    - Actually using ***Version #1*** *of the Cubic Formula* involves many complexities that could drive you crazy (as it did to with me!). I tried using it to solve these two equations:
      $2x^3 – 3x^2 + 8x – 12 = 0$ (which has solutions $^3/_2$, $\pm 2\boldsymbol{i}$)
      $3x^3 + x^2 – 10x – 8 = 0$ (which has solutions $-^4/_3$, -1, 2)
      You might be better off simply admiring the beauty of the formula, while maintaining your sanity.
    - I had more success using the (perhaps less elegant) ***Version #2*** *of the Cubic Formula*. If you wish, you could try doing one or two of the examples for yourself.
- **Puzzle! Three Shadows**
  1) A straight rod is embedded in a solid rectangular block made of translucent plastic. The sun is directly overhead. From a random corner, label the three faces which meet at that corner as A, B, C. When face A is placed on level ground the length of the shadow of the rod is 5 inches. When face B is placed on the ground the shadow's length is 6 inches, and when the face C is placed on the ground, the shadow's length is 7 inches. How long is the rod?

Formulas for Solving Polynomial Equations

Background

- In Lecture #1, I mentioned the *Cubic Formula* as one of the possible ways to solve our *Big Hanging Question*: “What are the three cube roots of 1?”
- What is the *Cubic Formula*? It is a formula (or a set of instructions) for solving a general cubic (polynomial) equation, such as $2x^3 - 3x^2 + 8x - 12 = 0$. (Note: the largest exponent is 3.)
- Let’s first list two much simpler formulas:

  The Linear Formula $\mathbf{x = -\frac{a}{b}}$

  Although you may have never seen (or needed) this formula, it can used to solve any linear equation $\mathbf{a\,x + b = 0}$ for real numbers a, b and $a \neq 0$.

  The Quadratic Formula $\mathbf{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$

  This well-known formula can used to solve any quadratic (second degree polynomial) equation $\mathbf{a\,x^2 + b\,x + c = 0}$ for real numbers a, b, c and $a \neq 0$.

- **The Cubic Formula**
  - *History*. Nearly 500 years ago, it was a tremendous accomplishment when a couple of Italian mathematicians had a (rather nasty) competition to find a *cubic formula* – with some success.
  - The general *cubic formula* is extremely complicated and can be expressed in multiple of very different ways. One way is listed below in a single equation (www.curtisbright.com).

$$x = \frac{-2b + \left(\frac{-1+\sqrt{-3}}{2}\right)^n \sqrt[3]{4\left(-2b^3 + 9abc - 27a^2d + \sqrt{(-2b^3 + 9abc - 27a^2d)^2 - 4(b^2 - 3ac)^3}\right)} + \left(\frac{-1-\sqrt{-3}}{2}\right)^n \sqrt[3]{4\left(-2b^3 + 9abc - 27a^2d - \sqrt{(-2b^3 + 9abc - 27a^2d)^2 - 4(b^2 - 3ac)^3}\right)}}{6a}$$

Although it is quite beautiful to see the whole formula in a single line, I have broken it into separate parts which makes the calculation more manageable. Even then, in most cases, using it to solve a cubic polynomial equation becomes horrifically complicated (usually two separate cube roots of (rather ugly) complex numbers).

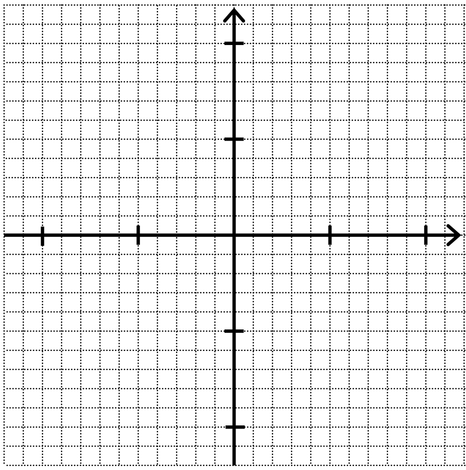
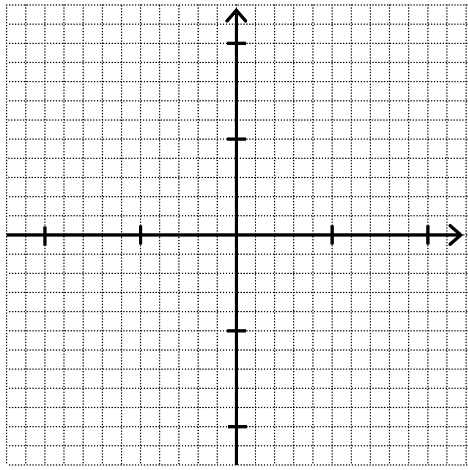
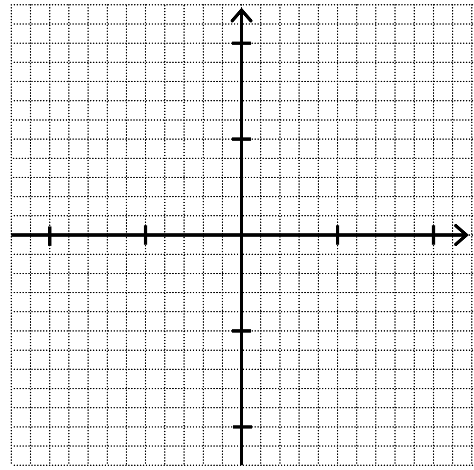
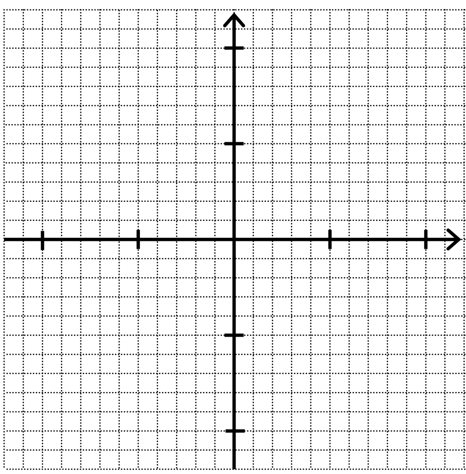
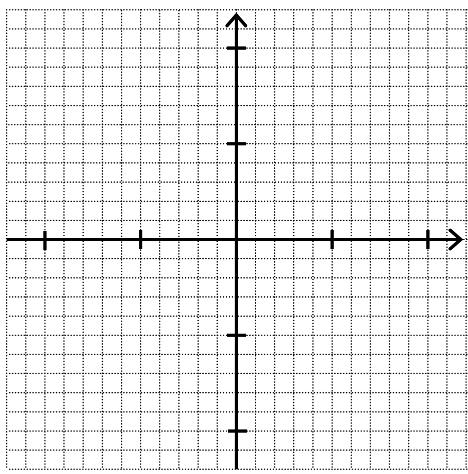
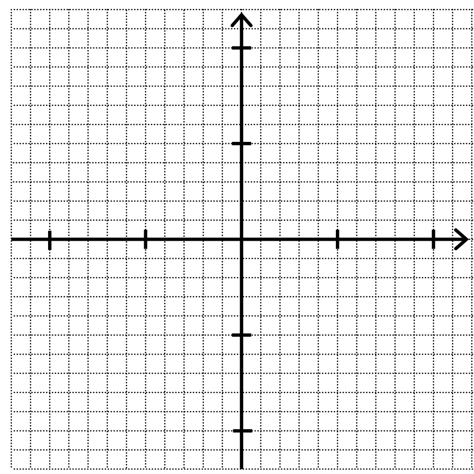
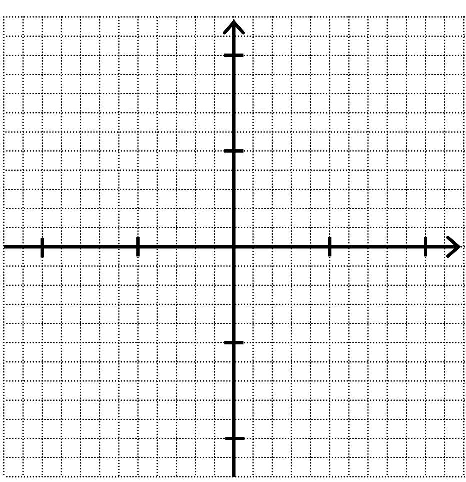
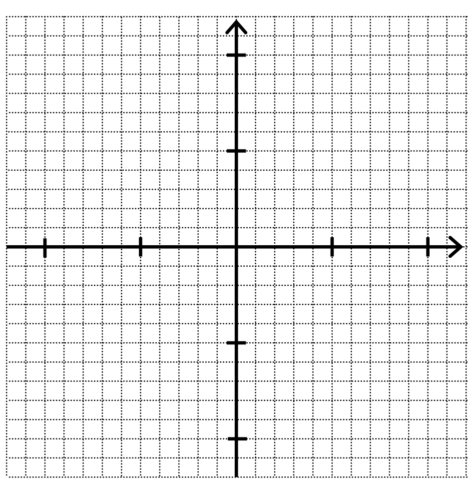
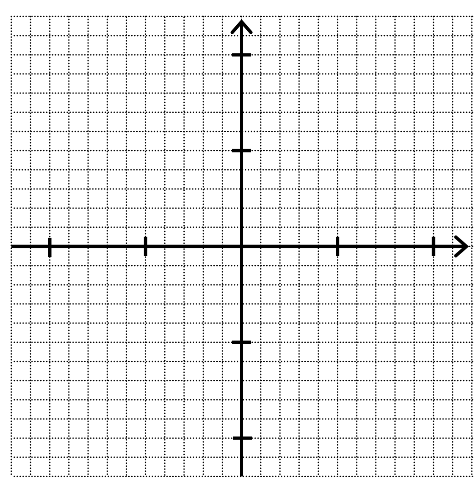
<u>The Cubic Formula</u> – Version #1 (a broken-down version of the above formula)

We are solving $\mathbf{a\,x^3 + b\,x^2 + c\,x + d = 0}$ for real numbers a, b, c, d and $a \neq 0$.

$\mathbf{P} = -2b^3 + 9abc - 27a^2d$; $\mathbf{Q} = b^2 - 3ac$; $\mathbf{R} = \sqrt{P^2 - 4Q^3}$; $\mathbf{D_1} = \frac{-1+\sqrt{3}}{2}$; $\mathbf{D_2} = \frac{-1-\sqrt{3}}{2}$

The solutions are (for n = 0, 1, 2) : $\mathbf{X_n} = \frac{-2b + \sqrt[3]{4(P+R)}\,D_1^n + \sqrt[3]{4(P-R)}\,D_2^n}{6a}$

- **The Quartic Formula** (for solving fourth degree polynomial equations).
  - The *quartic formula* (expressed in a single line) is around six times longer than the cubic formula! You can see it here: http://www.curtisbright.com/quartic/formulae-png.html

Problem Set #6

1) Find each of the roots.
(Give your answers in both rectangular and polar form.)
a) Two square roots of i.
b) Two square roots of 9i.
c) Four 4^{th} roots of –1.
d) Four 4^{th} roots of –16.
e) Three cube roots of –125.
f) Three cube roots of –8i
g) Three cube roots of $-4\sqrt{2} + 4\sqrt{2}\,i$.

2) What do you think happens if *i* is an exponent (e.g., 13^{i})?

Change of Base Formula for Exponents

We know that it is possible to change the base of a logarithm by using the following formula:

$$\log_a c = \frac{\log_b c}{\log_b a}$$

For example, if we are given $\log_4 300$, we can change the base to 10, which results in:

$$\log_4 300 \rightarrow \frac{\log_{10} 300}{\log_{10} 4}$$

Likewise, there are times when it is advantageous to change the base of an exponent. To do this, we use the following law of logarithms:

$$b^{\log_b n} = n$$

Taking both sides to the c gives us:

$$n^c = b^{c \cdot \log_b n},$$

which allows us to change the base of an exponent from n to b.

3) Change the base of…
a) 8^7 to base 2.
b) 8^7 to base 9.
c) $f(x) = 5^x$ to base 10.
d) $f(x) = 5^x$ to base e.
e) 9^{i} to base 3.
f) 13^{i} to base e.

Euler's Amazing Formula

Leonhard Euler discovered a formula that handles imaginary exponents. This famous formula (see our *High School Source Book* for a proof) is:

$$e^{ix} = \text{cis}(x),$$

where x is a positive real number given in radians.

4) Use Euler's formula in order to simplify each expression. Give your answer in rectangular form.

a) $e^{\frac{\pi}{4}i}$ c) $e^{\frac{7\pi}{6}i}$

b) $e^{\frac{\pi}{6}i}$ d) $e^{\frac{\pi}{2}i}$

e) $e^{i\pi}$ *Euler's Identity*

5) Explain how each of the following formulas can be derived from Euler's Formula.

a) $a^{i} = \text{cis}\left[\ln(a)\right]$
This shows that any positive real number, a, raised to the *i*, ends up on the unit circle.

b) $[\text{cis}(x)]^{i} = e^{-x}$
This shows that any complex number on the unit circle, raised to the *i*, results in a positive real number.

c) $[a \cdot \text{cis}(x)]^{i} = e^{-x}\,\text{cis}\left[\ln(a)\right]$
This formula is used to take any complex number to the *i*.

6) Simplify each expression.
a) 13^{i}
b) 7^{i}
c) $[\text{cis}(\pi/4)]^{i}$
d) $\left[\text{cis}\left(\frac{2\pi}{3}\right)\right]^{i}$
e) i^{i}
f) $[8 \cdot \text{cis}(\pi/6)]^{i}$
g) $[-3 + 3\sqrt{3}\,i]^{i}$

The Cubic Formula – Version #2

The general cubic equation: $x^3 + ax^2 + bx + c = 0$

We then substitute for x using: $x = (y - \frac{a}{3})$

Which simplifies to: $y^3 + \left(b - \frac{a^2}{3}\right)y + \left(\frac{2a^3}{27} - \frac{ab}{3} + c\right) = 0$

If we let: $p = b - \frac{a^2}{3}$ and $q = \frac{2a^3}{27} - \frac{ab}{3} + c$

Then we have an equation in the form of: $y^3 + py + q = 0$, (by Cardano, 1545.)

To solve this new equation, we let: $D = \left(\frac{1}{3}p\right)^3 + \left(\frac{1}{2}q\right)^2$

There are three possibilities:

If D > 0 then there is <u>one real solution</u>: $y = \sqrt[3]{-\frac{q}{2} + \sqrt{D}} + \sqrt[3]{-\frac{q}{2} - \sqrt{D}}$

If D = 0 then there are <u>two real solutions</u>: $y_1 = \mp 2\sqrt{-\frac{p}{3}}$; $y_2 = \pm\sqrt{-\frac{p}{3}}$

(y_2 is a double root) Use top sign if q is positive, bottom sign if q negative.

If D < 0 then there are <u>three real solutions</u>: Using $\alpha = \cos^{-1}\left|\frac{q}{2\sqrt{-(p/3)^3}}\right|$

$$y_1 = \mp 2\sqrt{-p/3}\,\cos(\alpha/3)$$
$$y_2 = \mp 2\sqrt{-p/3}\,\cos(\alpha/3 + 120°)$$
$$y_3 = \mp 2\sqrt{-p/3}\,\cos(\alpha/3 + 240°)$$

Use top sign if q is positive, bottom sign if q negative.

Example #1: Solve $x^3 - 2x^2 - 2x - 3 = 0$.

Solution: We first substitute $x = y + \frac{2}{3}$ (because $a = -2$), and we get

$p = -\frac{10}{3}$ and $q = -\frac{133}{27}$ which gives us the new equation $y^3 - \frac{10}{3}y - \frac{133}{27} = 0$.

We now get $D = \frac{169}{36}$ (or $4.69\overline{4}$), which is greater than zero, so using

$y = \sqrt[3]{-\frac{q}{2} + \sqrt{D}} + \sqrt[3]{-\frac{q}{2} - \sqrt{D}}$ we get $y = \frac{7}{3}$. Since $x = y + \frac{2}{3}$, our answer is <u>$x = 3$</u>.

(Curiously, the two complex roots are $-½ \pm \sqrt{3}/2$, the same that we found for the cube root of 1!)

Example #2: Solve $x^3 - 4x^2 + x + 6 = 0$.

Solution: We first substitute $x = y + \frac{4}{3}$ in for x and get

$p = -\frac{13}{3}$ and $q = \frac{70}{27}$ which gives us the new equation $y^3 - \frac{13}{3}y + \frac{70}{27} = 0$.

We now get $D = -1.\overline{3}$, which is less than zero, so we find $\alpha = 41.69366°$,

and our three solutions for y are $y_1 = -\frac{7}{3}$, $y_2 = \frac{5}{3}$, $y_3 = \frac{2}{3}$.

And, finally, using $x = y + \frac{4}{3}$, we get our final answers of $x_1 = -1$, $x_2 = 3$, $x_3 = 2$.

Example #3: Solve $x^3 + 9x^2 + 24x + 16 = 0$.

Solution: We substitute $x = y - 3$ and get $p = -3$, $q = -2$. Our new equation is $y^3 - 3y - 2 = 0$.

We now get $D = 0$, which means there are two solutions for y:

$y_1 = 2$, $y_2 = -1$, leading to our final solutions of $x_1 = -1$, $x_2 = -4$.