11^{th} Grade Assignment – Week #30

Individual Work

- From the unit *Complex Numbers – Part II*, choose problems that you need to practice from the following:
  - **Problem Set #1**: problems #1-36
  - **Problem Set #2**: problems #1-3
- Finish anything from the "Group Assignment" that your group doesn't complete.

Group Assignment:

for Tuesday

Discover some of the **Laws of Complex Numbers** by doing the following:

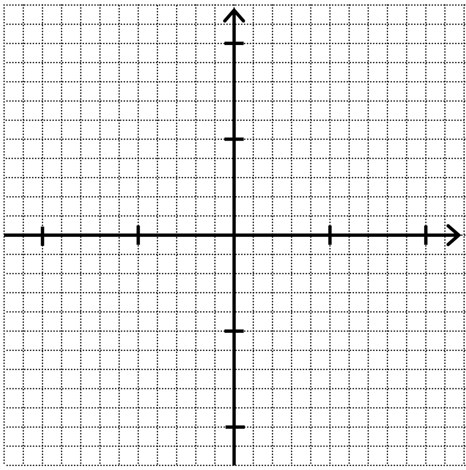
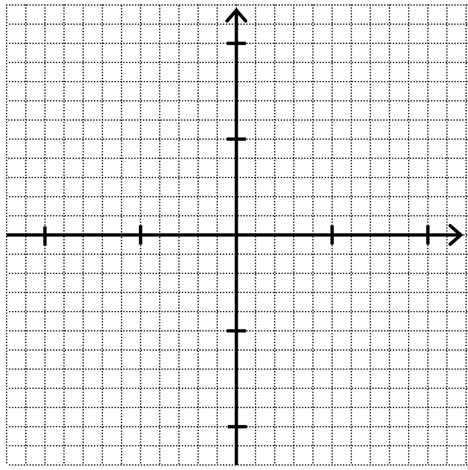
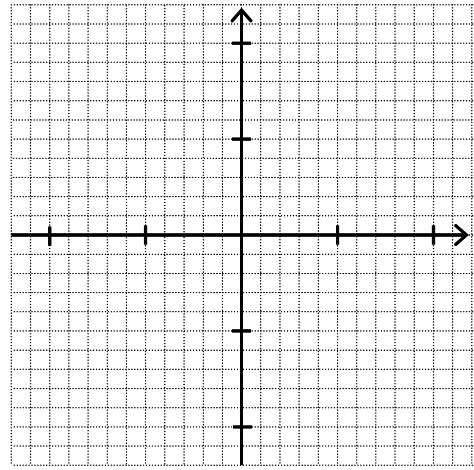
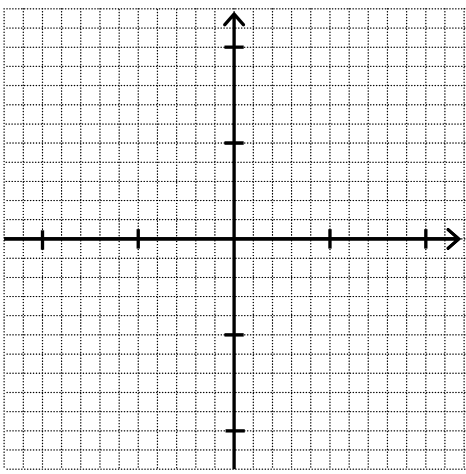
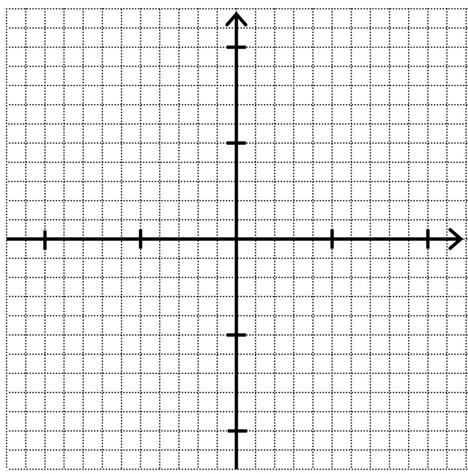
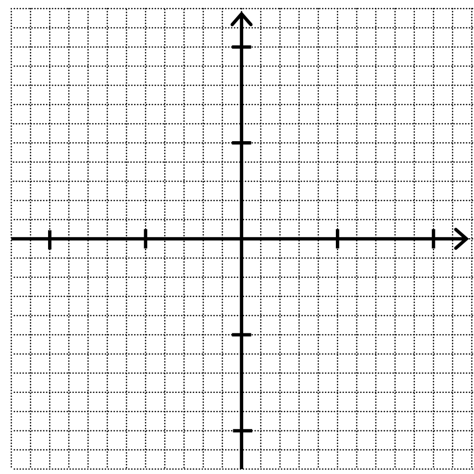
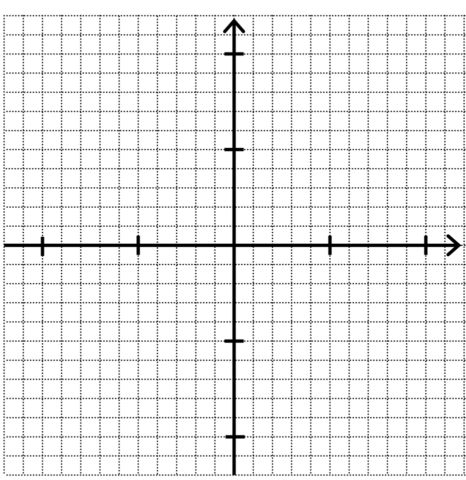
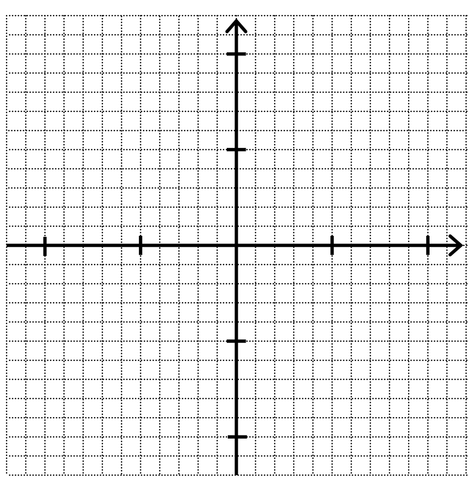
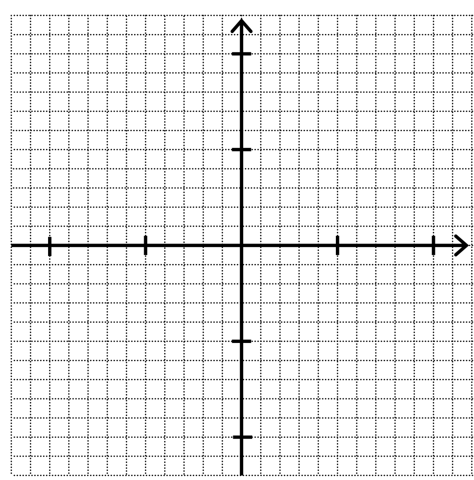
- **Problem Set #2**: problems #4-11
- **Problem Set #3**: All of it!
  (Note: The length of a vector is known as the "Absolute Value" (or "magnitude".))

for Thursday

- **Problem Set #4**: All of it!
- **Discovering a Change of Base Formula for Exponents**
  Make sure you understand everything below.
  - *This is the Change of Base Formula for Logarithms* $\log_a x = \frac{\log_b x}{\log_b a}$
  - This change of base formula allows us to change the base to something that is more convenient than what is given. For example, if we are to solve $\log_8 16$, then we can change into base 2 and instead solve $\frac{\log_2 16}{\log_2 8}$, which leads to an answer of $4/3$.
  - Here is the proof of the *Change of Base Formula for Logarithms:*

    Let $\log_a x = c$, which is also $a^c = x$
    Take the $\log_b$ of both sides:
    $\log_b (a^c) = \log_b x \rightarrow c\bullet\log_b a = \log_b x \rightarrow c = \frac{\log_b x}{\log_b a}$
    subbing back into the original equation gives us:

    $$\log_a x = \frac{\log_b x}{\log_b a}$$
  - Here is your task: derive the *Change of Base Formula for Exponents*, which would allow us to change the bas of an exponent without changing the value of the expression. For example, how could we change 7^x to an equivalent expression, but where the base is 3 instead of 7?

Complex Numbers – Part II

Problem Set #1

The Hanging Question

We are now close to our goal of answering the perplexing question: "*What are the three cube roots of 1*?"

But first, we'll need to review a few things…

Simplify.

1) $(4+i)(3+i)$
2) $(5+2i)(5-2i)$
3) $(5+2i)^2$
4) $(7+i)(7-i)$
5) i^2
6) i^3
7) i^4
8) i^{18}
9) $(3i)^2$
10) $(10i)^3$

11) $(7+5i)(6-2i)$
12) $(x-4)^2$
13) $(i-4)^2$
14) $(3+2i)^3$
15) $\frac{3}{i}$
16) $\frac{3}{4+i}$
17) $\frac{50i}{3-4i}$

"Complex" Factoring.
Each is possible!

18) $x^2 - 3$
19) $x^2 + 25$
20) $x^4 - 81$

Solve each equation over the set of complex numbers.

21) $x^2 + 13x + 40 = 0$
22) $x^2 + 12x + 40 = 0$
23) $x^2 - 3x + 6 = 0$
24) $x^2 - 3x - 6 = 0$

Trig Review.

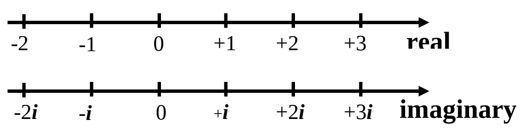
25) $\cos(\pi/3)$
26) $\sin(4\pi/3)$
27) $\tan(\pi)$
28) $\sec(\pi/4)$
29) $\csc(\pi/6)$
30) $\cot(\pi/3)$
31) $\cos(3\pi/2)$
32) $\sec(3\pi/2)$
33) $\csc(5\pi/3)$
34) $\tan(5\pi/3)$
35) $\sin(210°)$
36) $\sin(7\pi/6)$

Number line problems.
On one of the two number lines below graph each number.

37) 2
38) $-\frac{3}{4}$
39) π
40) $2i$
41) $-\frac{3}{4}i$
42) πi

Questions to think about.

- Is i greater or less than 1?
- How can we graph $3+5i$?
- We can graph the numbers $2^0, 2^1, 2^2, 2^3$, etc. all on the *real* number line. But what happens when we graph i^0, i^1, i^2, i^3, i^4, etc.?
- How can we "fix" these problems?

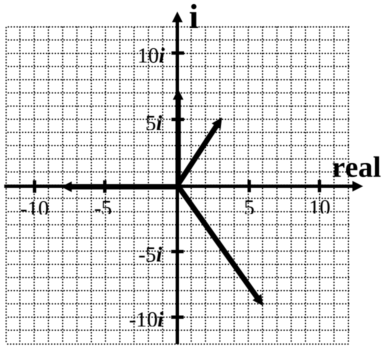


Problem Set #2

The Complex Number Plane

The questions posed at the end of the last problem set are addressed by combining the imaginary number line with the real number line. The result is the *complex plane*. Complex numbers are then graphed as vectors on this complex plane.

For example, the vectors for the numbers –8, 7*i*, 3+5*i*, and
6–9*i* are all shown below.



1) Graph each number as a vector on the complex plane.

a) 8	d) $-7-2i$
b) $-3i$	e) 0
c) $-4+6i$	f) $10i$

"Size" of a complex number.

The "size" of the complex number, a + b*i*, is known as its *absolute value*. Absolute value is the same as the length of its vector, which is $\sqrt{a^2+b^2}$.

2) Calculate the *absolute value* of each complex number.

a) $3-4i$	d) $-6-8i$
b) $2+i$	e) $7i$
c) $-7+9i$	f) -10

3) Simplify.

a) $(\sqrt{3}\,i)^4$

b) $\left(\frac{\sqrt{5}}{3}+\frac{1}{2}i\right)^2$

c) $(-\sqrt{3}+i)^3$

d) $\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)^4$

4) Graph a, b, c on one graph and f, g, h on another graph.

a = $5+3i$	**f** = $-3+5i$
b = $2+7i$	**g** = $8+6i$
c = **a** + **b**	**h** = **f** + **g**

5) What can be said about adding complex numbers?

6) Graph a, b on one graph and f, g on another graph.

a = $5+3i$	**f** = $3-2i$
b = 2•**a**	**g** = 4•**f**

7) What can be said about multiplying a complex number by a real number?

8) Graph a, b on one graph and f, g on another graph.

a = $5+3i$	**f** = $-8+2i$
b = *i*•**a**	**g** = *i*•**f**

9) What can be said about multiplying a complex number by *i*?

10) Graph a, b, c on one graph and f, g, h on another graph.

a = $1+2i$	**f** = $3+4i$
b = $2+3i$	**g** = $4+3i$
c = **a**•**b**	**h** = **f**•**g**

11) What can be said about multiplying two complex numbers?

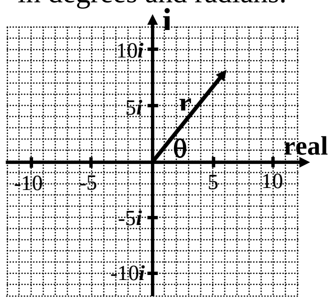
Problem Set #3

1) With the below graph…

a) What complex number is represented by the vector?

b) Find the absolute value (r) of the complex number.

c) Calculate the value of θ. Give your answer both in degrees and radians.



Rectangular & Polar Form

Writing a complex number (as we did above) as 6+8*i* is known as *rectangular form*. There is another form for writing complex numbers known as *polar form*.

In both cases we can imagine starting at the tail of the vector (the origin) and then following certain instructions in order to arrive at the head of the vector. In the case of a *Cartesian coordinate system* the head of the vector is given by the coordinates (6, 8*i*). A *polar coordinate system* gives that same point as (10, 53.1°), which means: "start at the tail, and move at an angle of 53.1° for a distance of 10."

The *polar form* of the above complex number is 10•cis(53.1°)

Of course, *polar form* may seem unnecessarily complicated, but soon you'll see how it makes certain things much simpler.

2) Convert from *polar form* into *rectangular form*.

a) 10•cis(45°) e) 6•cis(180°)

b) 10•cis($^{\pi}/_{4}$) f) cis(120°)

c) 2•cis(60°) g) 10•cis($^{3\pi}/_{2}$)

d) 7•cis(90°) h) 5•cis($^{5\pi}/_{3}$)

3) Convert from *rectangular form* into *polar form* (using radian measure).

a) $-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$ d) $-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i$

b) $\frac{\sqrt{3}}{2}+\frac{1}{2}i$ e) $2\sqrt{3}-2i$

c) $1+i$ f) $7i$

Some Laws for Complex Numbers

Look back at the last problem set in order to express laws (as formulas) for complex numbers.

Example: What is the law for adding two complex numbers?
Solution: This one is easier to express in *rectangular form*:
$(a_1+b_1i)+(a_2+b_2i) = (a_1+a_2)+(b_1+b_2)i$

Example: What is the law for multiplying a complex number by a real number, n?
Solution: Using *rectangular form*, we get:
n(a+b*i*) = (n•a)+(n•b)*i*
The same thing in *polar form* is:
n(r•cisθ) = (n•r)•cis(θ)
(The angle doesn't change!)

4) What is the law for multiplying a complex number by *i*? (Give answer both in polar form and rectangular form.)

5) What is the law for multi-plying two complex numbers? (Give answer both in polar form and rectangular form.)

Problem Set #4

1) Convert from *polar form* into *rectangular form*.

a) $8\cdot\text{cis}(\pi/6)$ d) $\frac{1}{3}\cdot\text{cis}\left(\frac{3\pi}{4}\right)$

b) $7\cdot\text{cis}(\pi/4)$ e) $\frac{1}{2}\cdot\text{cis}(300°)$

c) $5\cdot\text{cis}\left(\frac{3\pi}{2}\right)$ f) $\text{cis}\left(\frac{\pi}{2}\right)$

2) Convert from *rectangular form* into *polar form* (using radian measure).

a) $-\frac{1}{2}+\frac{\sqrt{3}}{2}i$ c) $10\sqrt{2}-10\sqrt{2}\,i$

b) $3+5i$ d) $-5-12i$

3) Looking at the above problems, and given that rectangular form is $a+bi$, and that polar form is $r\cdot\text{cis}\theta$.

a) What is the general formula for converting from polar form into rectangular form?

b) What is the general formula for converting from rectangular form into polar form?

Now use the formulas you have just derived.

4) Convert from *polar form* into *rectangular form*.

a) $6\cdot\text{cis}\left(\frac{7\pi}{6}\right)$ c) $3\cdot\text{cis}(5)$

b) $12\cdot\text{cis}(135°)$ d) $40\cdot\text{cis}(38°)$

5) Convert from *rectangular form* into *polar form*.

a) $7-7i$ b) $-4-7i$

6) For each value of **a**, determine the value of $\mathbf{a}^2$, calculate the absolute values of **a** and $\mathbf{a}^2$, and then graph
a and $\mathbf{a}^2$ on the same graph.

a) $\mathbf{a} = 4+3i$

b) $\mathbf{a} = 1+5i$

c) $\mathbf{a} = -\frac{5}{2}+6i$

7) What is the law for squaring a complex number? (Write this law both as a sentence and as a formula in polar form.)

8) Using $\mathbf{a} = 1+2i$, determine the values of $\mathbf{a}^2$, $\mathbf{a}^3$, and $\mathbf{a}^4$,
then calculate the absolute values of all four of these, and graph them all on the same graph.

9) **De Moivre's Theorem!**
What is the law for taking a complex number to the n^{th} power? (Write this law both as a sentence and as a formula in polar form.)

10) Simplify.

a) $\left(\sqrt{2}+\sqrt{2}\,i\right)^4$

b) $\left[2\cdot\text{cis}\left(\frac{\pi}{4}\right)\right]^4$

c) $\left(-5\sqrt{2}+5\sqrt{2}\,i\right)^3$

d) $\left[10\cdot\text{cis}\left(\frac{3\pi}{4}\right)\right]^3$

11) With the above problems, is polar form or rectangular form easier?