11^{th} Grade Assignment – Week #28

Individual Work

- From the *Logarithms* unit, do **Problem Sets #2 and #3.**

Group Assignment: *for Tuesday or Thursday.*

Look over all the problems first, and then decide which ones to work on.

1. *The Dartboard Problem (Part II)*.

Notes:

- If you do elect to do this problem, then you should do it on Tuesday because I will go over the solution in Wednesday's lecture (Lecture #2).
- These problems should only be attempted after completing *The Dartboard Problem – Part I* from last week.

Background. Make sure you understand all of this!

- With last week's *Dartboard Problem – Part I*, we found that the probabilities were as follows:

  Hitting the target in exactly one throw = $\frac{1}{12}$

  Hitting the target in exactly two throws = $\frac{11}{12}\cdot\frac{1}{12}$

  Hitting the target in exactly three throws = $\left(\frac{11}{12}\right)^2\cdot\frac{1}{12}$, etc.

- *Power Series Formula.* The group assignments from Week #15 and Week #26 both used the *Power Series Formula*: $\sum_{i=0}^{n} x^i = \frac{x^{n+1}-1}{x-1}$

  Example: $(1+4+4^2+4^3+4^4+4^5+4^6)$ is the same as $\sum_{i=0}^{6} 4^i$ and the formula gives us $\frac{4^7-1}{4-1} = 5461$.

- *A Variation of the Power Series Formula*

  In Week #15, the group assignment included this formula (and its proof):

$$\sum_{i=0}^{n}(i+1)x^i = \frac{(n+1)x^{n+2}-(n+2)x^{n+1}+1}{(x-1)^2}$$

- It is important to note that the following sequence : $1\cdot5^0+2\cdot5^1+3\cdot5^2+4\cdot5^3+5\cdot5^4+6\cdot5^5$

  can be written in sigma form as either $\sum_{i=0}^{5}(i+1)x^i$ or $\sum_{i=1}^{6} i\, x^{i-1}$

Here are the problems:

a) Use the above *Variation of the Power Series Formula* to evaluate the series: $1\cdot5^0+2\cdot5^1+3\cdot5^2+4\cdot5^3+5\cdot5^4+6\cdot5^5$.

b) What does the *Variation of the Power Series Formula* become in the case of an infinite series (where $n = \infty$) if it is known that $0 < x < 1$? $\sum_{i=0}^{\infty}(i+1)x^i$ for $0 < x < 1$

c) Give a formula for calculating the probability of hitting the target in exactly n throws.

d) **Putting it all together!** What is the *expected* (average) number of throws needed to finally hit the target? (Hint: Use ideas from everything above, and from the "bus problem" (#12g) in last Thursday's group assignment.)

e) *The Dartboard Principle*. What law can be stated about the above problem?

(More Problems on the Next Page →)

2. *Diophantus's Riddle*

The following is an ancient riddle about the Greek mathematician Diophantus.

> *Diophantus passed one sixth of his life in childhood, one twelfth in youth, and one seventh more as a bachelor. Five years after his marriage, his son was born. How old was Diophantus when he died if his son died four years earlier and lived to be half as old as his father did?*

3. *Two-Digit Numbers*

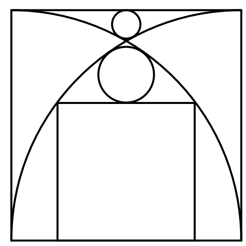
a) What two-digit number is twice the sum of its digits?

b) Two two-digit numbers are such that their digits are the reverse of one another and the ratio of the two (two-digit) numbers is 7:4. Find the numbers.

c) There are 2 two-digit numbers. The larger number is greater than 50, but the smaller number is not. The smaller number is a prime number, but the larger number is not. The sum of the two numbers is 10 less than twice the larger number. The sum of the second digits of both numbers is less than 11. What are the two numbers?

4. *Squares and Circles*

With the drawing shown here, the side of the outer square has a length equal to 1, and the two arcs are each a quarter of a circle. Find the length of the side of the smaller square, and the radii of the two circles.



Problem Set #2

Evaluate each without a calculator.

1) $\log_{20} 400$

2) $\log_{20}\left(\frac{1}{8000}\right)$

3) $\log_{20} 1$

4) $\log_{20}\left(\frac{1}{20}\right)$

5) $\log_{20} 20$

6) $\log_{20}(20^9)$

7) $\log_{20}(400 \cdot 8000)$

8) $20^{\log_{20} 53}$

Change of Base Formula.

9) State the Change of Base Formula.

10) How would you solve the following problems if you had a calculator that could only do logs in base 10?

a) $\log_8 100$

b) $\log_8 3$

c) $8^x = 100$

11) Is there a different Property of Logs you could use to solve 10c?

Solve for *x*. Use a calculator only if necessary.

12) $\log_3(9x) = 17$

13) $\log_3\left(\frac{27}{3^x}\right) = -10$

14) $e^x = 7$

15) $\log_x(64^4) = 4$

16) $1 - 4\log_7(3x + 1) = 5$

17) $6^{(x^2 + 2x + 3)} = 36$

18) $6^x = 216^{x-7}$

19) $36^x = 216^{x-7}$

20) $6^x = 150^{x-7}$

21) \$5000 is deposited into a savings account at 3.29% APR. Assuming no further activity on the account and the interest rate stays the same, how much money is in the account after three years if the interest is compounded:

a) Annually

b) Monthly

c) Daily

d) Every second

e) Continuously

f) Is there anything unexpected about your above answers?

22) There are two formulas that can be used for population growth:

Annual Growth: $\mathbf{P = P_0(1+r)^t}$

Continous Growth: $\mathbf{P = P_0 e^{rt}}$

The population of Chicago in 1990 was 2,783,726 and in 2000 was 2,896,016.

a) Find the growth rate, r, to 5 significant digits, using each of the two above formulas.

b) From the above problem, why are the values for r slightly different?

c) In what year will the population of Chicago reach 10,000,000?
(You can use either formula; you should get the same answer.)

d) What assumption are we making for the above problem?

Problem Set #3

How did you solve Problem #20 from the previous problem set? Below, we show two ways to solve a problem like this.

Example: Solve for x: $4^x = 3^{2x-1}$

Solution #1:

$\text{Log}_4 4^x = \log_4 3^{2x-1}$ $\log_4$ of both sides. We could also use another base.

$x \log_4 4 = (2x - 1)\log_4 3$ Prop. of logs.

$x = 2x \log_4 3 - \log_4 3$ Distribute.

$x - 2x \log_4 3 = -\log_4 3$ Subtract.

$x(1 - 2\log_4 3) = -\log_4 3$ Factor.

$x = \frac{-\log_4 3}{1 - 2\log_4 3} \approx 1.3548$ Divide.

or

Solution #2:

Convert 4 into a power of 3

(or convert 3 into a power of 4):

$4 = 3^Q \rightarrow Q = 1.26186$

Therefore $4^x = 3^{2x-1}$ becomes

$3^{1.26186 \cdot x} \approx 3^{2x-1}$

$1.26186 \cdot x \approx 2x - 1$

$1 \approx 0.73814 \cdot x$

$x \approx 1.3548$

Solve for x.

1) $7^x = 40$
2) $3^{5-x} = 27$
3) $3^{5-x} = 20$
4) $8^x = 4^{8x+2}$
5) $8^x = 13^{8x+2}$
6) $8(10^{3x}) = 12$
7) $e^{2x+1} = 9$
8) $x^2 - 3x - 4 = 0$
9) $e^{2x} - 3e^x - 4 = 0$
10) $\log_{10}(x + 15) - \log_{10}x = \log_{10}(x + 3)$
11) $\left(1 + \frac{0.0325}{12}\right)^{12x} = 3$
12) $10,000 is deposited into a money-market account and doubles in five years. What was the APR on the account during this time given that the interest was compounded:
    a) Monthly
    b) Continuously
13) The population of the metropolitan area of São Paulo, Brazil was about
    2.4 million in 1950 and
    4.7 million in 1960.
    a) What was São Paulo's growth rate during the 1950's? (You can give your answer as an annual growth rate, or as a continuous growth rate.)

    Based on that growth rate, what would the population of São Paulo be…

    b) in 1980?
    c) in 2020?
    d) São Paulo's population growth rate from the 1950's was extreme. How long do you think that a city can grow at that rate?
14) A new car costs $18,599 and after two years it is worth $11,500. At that rate of depreciation…
    a) How much is the car worth after three years?
    b) How long before the car is worth $2000?