11th Grade Assignment – Week #15

Announcements

- There is an extra lecture (lecture #3) this week, which helps to explain some of the problems from the (very challenging) group assignment. You should only watch this video after your Tuesday group meeting where you have tried some of these problems.

(Repeat) Announcement regarding the upcoming Projective Geometry Main lesson

- It is a 3-week main lesson, running from Week #16 to Week #18.
- Each day you will watch about 40 minutes of recorded lectures, which includes drawing instructions.
- You should read the "**Projective Geom Expectations and Schedule**" document, which can be accessed through the Download/Assignment Page, the link for which is found in your "Welcome Email".
- The only live lecture during those three weeks is the usual Friday tutorial (at the usual time).
- As opposed to the rest of your Math Academy experience this year, this main lesson does not have assignments designed for group work. However, if you wish to meet together to discuss some of the assignments, you are certainly welcome to do so.
- Part of the advantage with how this main lesson is set up is that you have added flexibility. For example, if you have a very busy day with other things, you can skip a day of doing projective geometry, and then catch up with your projective geometry work on another day.

Individual Work

- Select problems that you need to work on from **Problem Set #6** (Cartesian Geometry – Part II).
- **Take the test** (found at the end of this document) before the start of week #16.

Group Assignment: *For Tuesday or Thursday*

1) **Age Puzzle.** The product of Stacy's, Tracy's and Lacy's ages is 240. Two years ago, Stacy was twice Lacy's age. How old are they now? (Give all possible answers.)

2) **Series.** Start by reading (and understanding!) the following background:

A series is created when you add up the terms in a sequence.

Example #1: Instead of writing 1+2+3+4+5+6, I can write it in series Σ notation as $\sum_{i=1}^{6} i$ It may be helpful to think of this as a "loop", where i is first equal to 1, then 2 then 3, all the way up to 6, which is the number on top of the Σ. We then add everything together to get a final answer of 21.

(Note that "i" has nothng to with "imaginary", and Σ is the Greek letter *sigma*.)

Example #2: Evaluate $\sum_{i=8}^{11} 5i$. Normally i starts at either 0 or 1, but in this case (for no apparent reason) i starts at 8 and goes to 11. When i is 8, the result is 40 (which is 5x8), and then when i is 9 we get 45, etc. Therefore this series could be rewritten as 40+45+50+55, which yields a final answer of 190.

Example #3: Let's add together the even numbers from 50 to 60, which is the (finite) sequence: 50, 52, 54, 56, 58, 60. We can write this as: 50+52+54+56+58+60 or we can use Σ notation: $\sum_{i=1}^{6} 48+2i$, which simply says that we need to add together the first six terms of the given series. The first term of the series is found by putting in i = 1 into 48+2i (which gives us 50), then we put in i=2 into 48+2i (which gives us 52), and so forth up to i = 6. In the end we can "evaluate" that the series is equal to:

$\sum_{i=1}^{6} 48+2i$ = 50+52+54+56+58+60 = **330**

Important note: We could have also written the series as $\sum_{k=1}^{6} 48+2k$ or $\sum_{i=0}^{5} 50+2i$, or in a number of other ways. Either way, the series will be equal to 330.

(See Next Page →)

(Group Assignment, Continued.)

2) Evaluate each series:

A. $\sum_{i=0}^{5} i$

B. $\sum_{i=1}^{6} i-1$

C. $\sum_{i=0}^{4} 3i+2$

D. $\sum_{i=1}^{100} i$

E. $\sum_{i=1}^{6} i^2$

F. $\sum_{i=0}^{5} 3$

G. $\sum_{i=1}^{6} 3$

H. $\sum_{i=1}^{n} 3$

I. $\sum_{i=1}^{n} b$

3) **More (challenging) Power Series**

A. Show that the following identity/formula is true: $\sum_{i=0}^{n} x^i = \frac{x^{n+1}-1}{x-1}$

(Hint: $1 + x + x^2 + x^3 + x^4$ can be written as $\sum_{i=0}^{4} x^i$

B. What does the above identity become in the case of an infinite series (where $n = \infty$) if it is known that $0 < x < 1$? Use this to evaluate: $\sum_{i=0}^{\infty} (⅓)^i$

C. **Study the below proof** for an identity/formula for the series: $\sum_{i=0}^{n} (i+1)x^i$

The given series is $1 + 2x + 3x^2 + 4x^3 + \ldots + (n+1)x^n$.

$$1 + x + x^2 + x^3 + \ldots + x^n = \frac{x^{n+1}-1}{x-1}$$ (This comes from Part A, above)

$$x + x^2 + x^3 + \ldots + x^n = \frac{x^{n+1}-x}{x-1}$$ (See ** below for explanation.)

$$x^2 + x^3 + \ldots + x^n = \frac{x^{n+1}-x^2}{x-1}$$

$$x^3 + \ldots + x^n = \frac{x^{n+1}-x^3}{x-1}$$

etc., until…

$$x^n = \frac{x^{n+1}-x^n}{x-1}$$

Now, adding up all of the above equations, we get:

$$1 + 2x + 3x^2 + 4x^3 + \ldots + (n+1)x^n = \frac{(n+1)x^{n+1} - (1+x+x^2+\ldots+x^n)}{x-1}$$

Which eventually simplifies to:

$$1 + 2x + 3x^2 + 4x^3 + \ldots + (n+1)x^n = \frac{(n+1)x^{n+2} - (n+2)x^{n+1} + 1}{(x-1)^2}$$

$$\sum_{i=0}^{n} (i+1)x^i = \frac{(n+1)x^{n+2} - (n+2)x^{n+1} + 1}{(x-1)^2}$$

D. Use the above identity/formula to evaluate the series: $1\cdot4^0 + 2\cdot4^1 + 3\cdot4^2 + 4\cdot4^3 + 5\cdot4^4 + 6\cdot4^5 + 7\cdot4^6$.

** $x + x^2 + x^3 + \ldots + x^n \rightarrow x(1 + x + x^2 + x^3 + \ldots + x^{n-1}) \rightarrow x\left(\frac{x^n-1}{x-1}\right) \rightarrow \frac{x^{n+1}-x}{x-1}$

Problem Set #6

1) Graph each of the following.

a) $3x - 2y = 4$

b) $y = 2x^2 - 4$

c) $x^2 + (y-4)^2 = 2$

d) $\frac{x^2}{25} + \frac{y^2}{4} = 1$

e) $4x^2 + 25y^2 = 100$

f) $(x-8)^2 + \frac{(y+5)^2}{4} = 1$

g) $y = -(x-5)^2 - 3$

h) $y = -\frac{1}{3}(x+7)^2$

i) $(x+9)^2 + (y+3)^2 = 9$

j) $y = x^2 - 8x + 14$

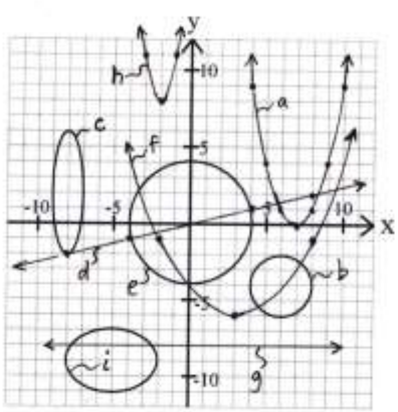
k) $2x^2 + y^2 - 12x + 10y + 41 = 0$

l) $x^2 + 8x + y^2 + 25 = 0$

m) $y = -\frac{1}{4}x^2 + x + 3$

n) $\frac{4(x+1)^2}{3} + \frac{(y-4)^2}{9} = 1$

2) Give the equation of each of the following graphs.



3) Graph each of the two given equations and then find the common solution (if approximate, then give it to three significant digits).

a) $3x + 5y = 9$
$2x - y = 6$

b) $4x - 2y + 10 = 0$
$y - x^2 - 10x - 17 = 0$

c) $y = x^2 - 2$
$9x^2 + 25y^2 = 225$

d) $x - y = 2$
$(x-5)^2 + (y-1)^2 = 4$

4) Graph the below equations and label the (real) roots of each equation with the exact coordinates.

a) $y = \frac{2}{3}x - 4$

b) $(x-2)^2 + (y+2)^2 = 9$

c) $y = (x+4)^2 + 7$

5) Graph the below equation and then give the x-intercepts and y-intercepts.

$$y = (x+1)^2 - 9$$

6) Give three solutions to the above equation.

7) Consider these two sentences: *The sum of two numbers is 14. The sum of their squares is 100.*

a) Give two-variable equations (using x and y) that express each of the two sentences.

b) Graph these two equations.

c) Find the common solution to the two equations.

Cartesian Geometry II Test

All Problems are worth 4 points.

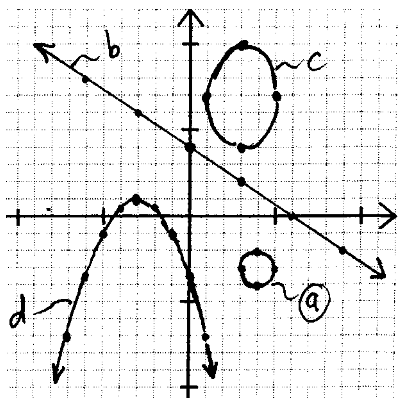
1) Give the equation of each graph, given below.

a)

b)

c)

d)



2) Give two solutions to the equation you gave as an answer to #1d, above.

3) Graph each of the following on the graphs below. Make sure you label each one.

a) $y = -2(x-1)^2 + 5$

b) $x - 3y = 12$

c) $x^2 + (y+2)^2 = 4$

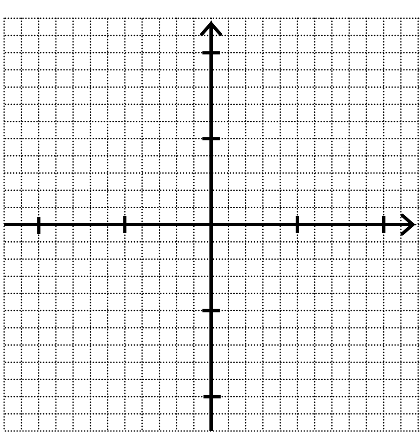
d) $\frac{x^2}{25} + \frac{y^2}{9} = 1$

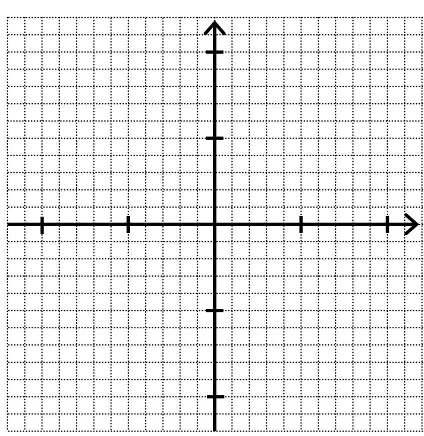
e) $x = -y^2 + 3$

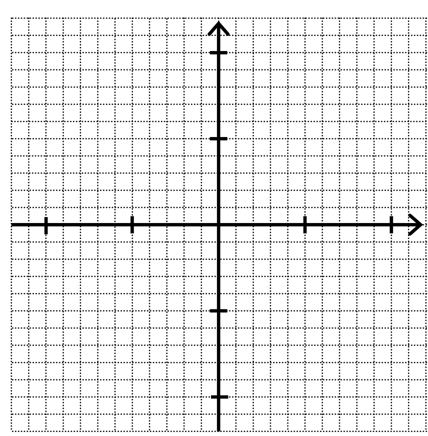
f) $25x^2 + 4y^2 = 100$

g) $y^2 + 10y + 17 = 2x - x^2$

(Please turn over→)







4) Find the exact common solution to the two equations.
$x - y = 2$
$(x-5)^2 + (y-1)^2 = 4$

5) Using the equation
$x^2 + y^2 - 6x + 4y + 4 = 0$
Where is $y = -5$?

6) Give the x-intercepts y-intercepts, and roots for:
$y = (x+1)^2 - 9$

7) *Challenge!* (2 points extra credit.)
graph $20y^2 = 30x - 5x^2$