11^{th} Grade Assignment – Week #11

Group Assignment:

For Tuesday

1) On Saturday, starting at 7am, Bob the monk hiked four miles to a temple at the top of a mountain. After spending the rest of the day and the next night at the temple, he woke the next morning (Sunday), and, at 7am, started his hike back down the mountain, along the same trail as the previous day. The ratio of his walking speed uphill to his speed downhill is 2:5. Both speeds were constant. How far from the temple was he when he was at the exact same place at the same time on the two different days?

For Thursday

Note: With the below "Crazy Factoring" problem, we are operating under the assumption that x can only be *real numbers*. In that case, a polynomial like $x^2 + 1$ cannot be factored. This is in contrast to the problems we are working on in the workbook (in our *Individual Work*) where we are factoring over the set of *complex numbers*, in which case a polynomial like $x^2 + 9$ can be factored to $(x + 3i)(x - 3i)$. Going back and forth between allowing for complex numbers and limiting ourselves to real numbers, is a fairly common practice in mathematics.

- *Crazy Factoring – Part II*

  Here are some laws about factoring polynomials that you may have discovered last time:
  - If n is odd then $(x+1)$ is a factor of $(x^n + 1)$.
  - For any n $(x-1)$ is a factor of $(x^n - 1)$.
  - If i is a factor of n then $(x^i - 1)$ is a factor of $(x^n - 1)$.
    (For example, $x^7 - 1$ must be a factor of $x^{21} - 1$.)
  - $a^3 \pm b^3 \equiv (a^2 \mp ab+b^2)(a \pm b)$ (You saw this last week.)

  The above laws may help you with the following two questions:

  2) Completely factor the polynomial $x^{15} - 1$
     Hints: Polynomial long division may be helpful, and x^2+x+1 divides evenly into $x^{10}+x^5+1$
  3) Determine the prime factorization of $2^{60} - 1$.
     Notes:
     - $2^{60} - 1 = 1{,}152{,}921{,}504{,}606{,}846{,}975$ (But you don't actually need to know that!)
     - The above laws for factoring polynomials are also helpful for this problem.
     - You may use a calculator and a table of prime numbers, found at the end of this document.
     - To determine if a large number is prime, you only need to try dividing it by all the prime numbers up until the square root of the large number.
     - 1047553 is divisible by 1321, which is a prime number.

Individual Work

- Carefully select problems that are helpful for you from **Problem Sets #4-5** (from *Complex Numbers, Part I*).

Problem Set #4

1) Let $c = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i$

On the previous problem set we found that $c^2 = i$ and $c^8 = 1$. Now find c^3, c^4, c^5, c^6, c^7.

Simplify.

2) $(4i)(3i)$

3) $\sqrt{-20} \cdot \sqrt{-5}$

4) $(2 - i)(5 + i)$

5) $(6 + 3i)(2 + i)$

6) $(4 - 3i)(7 + 3i)$

7) $(4 - 3i)(4 + 3i)$

8) $(3 - 8i)(3 + 8i)$

9) $(3 - 8i)(3 - 8i)$

10) $(2 - i)^2$

11) $(3 + 5i)^2$

12) $(10 + i)^3$

An Important Theorem

In 1799, Carl Friedrich Gauss proved, at the age of 22, an important theorem that is now called *The Fundamental Theorem of Algebra*. One of the important consequences is that:

An n^{th} degree polynomial equation will always have n roots (some roots may repeat).

Example: Solve $x^2 + 5x - 6 = 0$.
Solution: $x = -6, 1$

Example: Solve $x^2 - 6x + 9 = 0$
Solution: $x = 3$ (double root!)

Example: Solve $x^2 - 5 = 0$
Solution: $x = \pm\sqrt{5}$

Example: Solve $x^2 + 13 = 0$
Solution: $x = \pm\sqrt{13}\, i$

Example: Factor $x^4 - 3x^2 - 28$
Sol: $(x+\sqrt{7})(x-\sqrt{7})(x+2i)(x-2i)$

"Complex" Factoring. (All are possible!)

13) $x^2 - 9$

14) $x^2 + 9$

15) $x^2 - 11$

16) $x^2 + 5$

17) $x^4 - 1$

Solve. (For the rest of this unit all complex number solutions should be considered.)

18) $x^2 - 16 = 0$

19) $x^2 + 16 = 0$

20) $x^2 - 11 = 0$

21) $x^2 + 11 = 0$

22) $x^2 - 6x + 25 = 0$

23) $x^2 + 3x + 5 = 0$

24) $x^2 + 5x - 14 = 0$

25) $x^2 + 5x + 14 = 0$

26) $x^4 + 4x^2 - 5 = 0$

27) $x^4 + 7x^2 + 12 = 0$

28) $x^2 + 5x - 8 = 0$

29) $x^2 + 5x + 8 = 0$

30) $3x^2 + 5x + 14 = 0$

31) $(x - 2)^3 = 13 + 3(x - 7)$

Given these function definitions:

$f(x) = 3x^2$

$g(x) = 2x - 7$

Find each value.

32) $f(10)$

33) $g(10)$

34) $f(g(10))$

35) $g(f(10))$

36) $f(g(x))$

37) $g(f(x))$

38) $g(f(x + 3))$

Problem Set #5

Simplify.

1) $3x^4(5x^2)$
2) $3x^4(x^2 + 5)^2$
3) $(7 + x)(4 + x)$
4) $(7 + i)(4 + i)$
5) $(7 + \sqrt{3})(4 + \sqrt{3})$
6) $(5 - i)(3 + i)$
7) $(2 + 3i)(6 + 5i)$
8) $(x - 7)^2$
9) $(7 - i)^2$
10) $(x + 4)^3$
11) $(4 + i)^3$
12) $[x + (4+3i)][x + (4-3i)]$
13) $[x - (4+3i)][x - (4-3i)]$
14) $\frac{4}{6 + \sqrt{2}}$
15) $\frac{3 + \sqrt{5}}{2 + \sqrt{3}}$
16) $\frac{3}{2 + i}$
17) $\frac{5 + i}{3 + i}$
18) $\frac{3 + 5i}{1 - 4i}$

"Complex" Factoring.

19) $x^2 - 400$
20) $x^2 - 200$
21) $x^2 + 400$
22) $x^2 + 200$
23) $x^3 + 4x$
24) Factor $x^2 - 8x + 25$

25) Where else can you find the answer to the above problem?

Interesting Questions

Answer, or think about, the below questions. Some of them will remain unanswered for a while. (Recall that an n^{th} degree polynomial equation will always have n roots.)

26) Find the two square roots...
- a) of 9.
  (The answer to this question is the solution to $x^2 = 9$.)
- b) of 7.
- c) of –4.
- d) of –5.
- e) of *i*. (Hint: Look at set #3!)
- f) of –*i*.

27) Find the four 4^{th} roots…
- a) of 1.
- b) of 16.
- c) of –1.

28) What are the eight 8^{th} roots of 1? (See the previous problem set.)

29) What are the three cube roots of 1?

Prime Numbers up to 2000 (in groups of 250)

2	251	503	751	1009	1259	1511	1753
3	257	509	757	1013	1277	1523	1759
5	263	521	761	1019	1279	1531	1777
7	269	523	769	1021	1283	1543	1783
11	271	541	773	1031	1289	1549	1787
13	277	547	787	1033	1291	1553	1789
17	281	557	797	1039	1297	1559	1801
19	283	563	809	1049	1301	1567	1811
23	293	569	811	1051	1303	1571	1823
29	307	571	821	1061	1307	1579	1831
31	311	577	823	1063	1319	1583	1847
37	313	587	827	1069	1321	1597	1861
41	317	593	829	1087	1327	1601	1867
43	331	599	839	1091	1361	1607	1871
47	337	601	853	1093	1367	1609	1873
53	347	607	857	1097	1373	1613	1877
59	349	613	859	1103	1381	1619	1879
61	353	617	863	1109	1399	1621	1889
67	359	619	877	1117	1409	1627	1901
71	367	631	881	1123	1423	1637	1907
73	373	641	883	1129	1427	1657	1913
79	379	643	887	1151	1429	1663	1931
83	383	647	907	1153	1433	1667	1933
89	389	653	911	1163	1439	1669	1949
97	397	659	919	1171	1447	1693	1951
101	401	661	929	1181	1451	1697	1973
103	409	673	937	1187	1453	1699	1979
107	419	677	941	1193	1459	1709	1987
109	421	683	947	1201	1471	1721	1993
113	431	691	953	1213	1481	1723	1997
127	433	701	967	1217	1483	1733	1999
131	439	709	971	1223	1487	1741	
137	443	719	977	1229	1489	1747	
139	449	727	983	1231	1493		
149	457	733	991	1237	1499		
151	461	739	997	1249			
157	463	743					
163	467						
167	479						
173	487						
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