11^{th} Grade Assignment – Week #10

Group Assignment:

For Tuesday

- *Perfect Numbers.* A perfect number is a number that is the sum of its "proper" factors (not including the number itself). For example, 28 is a perfect number because its "proper" factors are 1, 2, 4, 7, 14, and that adds to 28. 6 is also a perfect number because its proper factors are 1,2,3, which adds to 6. Perfect numbers were of great interest to the Greeks, but they only knew the first four of them: 6, 28, 496, 8128. In spite of great efforts, the fifth perfect number remained undiscovered for over 1500 years. **Determine the fifth perfect number**.

For Thursday

Note: With the below "Crazy Factoring" problem, we are operating under the assumption that x can only be *real numbers*. In that case, a polynomial like x^2+1 cannot be factored. This is in contrast to the problems we are working on in the workbook (in our *Individual Work*) where we are factoring over the set of *complex numbers,* in which case a polynomial like x^2+9 can be factored to $(x+3i)\,(x-3i)$. Going back and forth between allowing for complex numbers and limiting ourselves to real numbers, is a fairly common practice in mathematics.

- *Crazy Factoring! (Part I)*

  Background: Make sure everyone in your group understands everything below.
  - If you multiply out $(x^2+5x+2)(x+3)$ you get $x^3+8x^2+17x+6$
    Although factoring a cubic (exponent of 3) polynomial is difficult, we can reverse the above statement, and say that $x^3+8x^2+17x+6$ factors to $(x^2+5x+2)(x+3)$
  - Similarly, but surprisingly, if you multiply out $(x^2-5x+25)(x+5)$ you just get x^3+125
    (You should multiply this out for yourself to show that it is correct!)
    Therefore, we can say that x^3+125 factors to $(x^2-5x+25)(x+5)$.
    It turns out that x^3-125 factors to $(x^2+5x+25)(x-5)$
  - Both of the above can be combined and written as $x^3\pm 5^3$ factors to $(x^2\mp 5x+5^2)(x\pm 5)$
    This law can be written more generally as this identity: $a^3\pm b^3\equiv(a^2\mp ab+b^2)(a\pm b)$
    This is known as factoring the sum and difference of two cubes.
  - This can even be used to factor binomials with exponents that are multiples of 3.
    Example: $x^{15}+27 \rightarrow (x^5)^3+3^3 \rightarrow (x^{10}-3x^5+9)(x^5+3)$

  Your task: Using the binomial expression $x^{12}-1$ do these two things:
  1) Factor it completely.
  2) List all of its binomial factors.

  Hints:
  - A binomial factor could be something like x^5-7 (which in this case is not correct).
  - One of the binomial factors of $x^{12}-1$ is x^6+1.
  - There are a total of 9 binomial factors of $x^{12}-1$
  - For the purpose of this exercise, we will only use whole number coefficients
    (no complex/imaginary numbers).

  Also: What laws can you find that are related to this?

Individual Work

- Carefully select problems that are helpful for you from **Problem Sets #1-3** (from *Complex Numbers, Part I*). A few things to keep in mind with these problem sets:
  - *Functions and Function Notation*. This is introduced right at the start of Problem Set #1, but I don't mention functions in the lectures until the second lecture. If you have never seen functions before, you may wish to wait until you have seen this week's second lecture.
  - *Algebra Review*. Problems Sets #1 and #2 have a fair bit of review of some basic algebra from 9^{th} grade, with a focus on factoring. If you are rusty with these topics, this review may be quite helpful.

Complex Numbers – Part I

Problem Set #1

Function Notation

At this point, we are comfortable with seeing a mathematical relation expressed between x and y, such as:

$y = x^2 + 3x - 8$

We can then ask a question like, "*What is the value of y when x equals 4*?"

In higher level mathematics, this same relation is often expressed in *function notation* as:

$f(x) = x^2 + 3x - 8$

"f(x)" should be read as "f of x".

And the same question can then be expressed simply as "*Find f(x)*".

As with learning any new language, the language of functions can at first seem rather strange, but after a short while it will seem like second nature.

Evaluate each function:

Example: Find f(3) given that $f(x) = 5x - 7$

Solution: We simply put 3 into x on the right side of the equals sign, and get $5 \cdot 3 - 7$.
Therefore f(3) = 8.

1) Find f(5) given that $f(x) = x^2 - 6$.

2) Find f(2) given that $f(x) = 3x^2 + 4x + 1$.

3) Find f(16) given that $f(x) = 2x + \sqrt{x}$

4) Find g(–3) given that $g(x) = 4x + 8$

5) Find h(0) given that $h(y) = y^2 - 7y + 3$

Review

Factor.

6) $x^2 + 10x + 24$

7) $x^2 + 10x - 24$

8) $x^2 - 10x + 24$

9) $x^2 - 10x - 24$

10) $x^2 + 19x + 60$

11) $3x^2 + 8x + 5$

12) $x^2 - 36$

13) $x^6 - 4$

14) $x^8 + 9$

Solve for x.

15) $7 + 2(6x - 3) = 8x - 1 + x$

16) $\frac{2}{5}x - 3 = \frac{3}{10}$

17) $\frac{2}{5}x - 3 = \frac{3}{10}x$

18) $\frac{4}{7}x = \frac{5}{6}$

19) $\frac{4}{7} + x = \frac{5}{6}$

20) $x^2 - 16 = 0$

21) $x^2 - 5 = 0$

22) $x^2 + 5 = 0$

Problem Set #2

1) Compare the last two problems from the previous problem set. Why was one possible and the other wasn't?

Given these function definitions:

$f(x) = 3x^2 - 4x + 8$

$g(x) = 5x - 3$

$h(x) = x^2$

Find each value.

2) $f(10)$
3) $g(10)$
4) $h(10)$
5) $f(-3)$
6) $g(\frac{1}{2})$
7) $h(-7)$
8) $f(0)$
9) $g(0)$
10) $h(\frac{1}{3})$
11) $f(\sqrt{5})$
12) $g(\sqrt{5})$
13) $h(\sqrt{17})$
14) $h(\sqrt{5})$
15) $h(\sqrt{-5})$

Review

Factor.

16) $x^2 + 7x - 8$
17) $x^2 - 13x + 30$
18) $x^2 + 13x - 30$
19) $x^2 + 13x + 30$
20) $x^2 - 13x - 30$
21) $4x^2 - 3x - 10$
22) $15x^3 - 10x$
23) $x^4 - 16$
24) $3x^5 - 12x^3$
25) $20x^5 - 60x^4 + 40x^3$

Solve for x.

26) $\frac{4}{7}x = 2\frac{5}{6}$
27) $x - \frac{2}{3} = 2\frac{5}{6}$
28) $\frac{2x+3}{7} = \frac{x-5}{4}$
29) $\frac{1}{4}x - 3 = \frac{3}{4} - \frac{1}{2}(3x + \frac{2}{3})$
30) $x^2 + 5x = 5x + 16$
31) $2x^2 + 8x = x^2 + 3x + 36$

Multiple Roots

32) Consider the equation $x^2 = 1$. How many *roots* are there to this equation? ("Root" is another word for solution.) (We also could have asked, "How many square roots of 1 are there?")

33) Why are there that many roots to the above equation?

34) How many roots should there be for $x^4 = 1$? (i.e., "How many fourth roots of 1 are there?")

A New Twist to Factoring

We know that one way to solve the equation $x^2 - 16 = 0$ is to first factor it to $(x+4)(x-4) = 0$.

We will now factor some expressions that previously we said were impossible.

Example: Solve $x^2 - 7 = 0$

Solution: $(x + ?)(x - ?) = 0$

"?" would have to be $\sqrt{7}$ to make this work, so the solution is $x = \pm\sqrt{7}$.

Example: Solve $x^2 + 9 = 0$.

Solution: Using the method shown in the above example, we factor it to

$(x + ?)(x - ?) = 0$

What would "?" have to be equal to in order to make it work?

Two Perplexing Questions!

We are now left with two perplexing questions:

- What are the 4 fourth roots of 1?
- How can we factor $x^2 + 9$?

Problem Set #3

A Brief History of Numbers

In mathematics, there have been many times when people said that something was impossible (or not permitted) and then this barrier was broken and a new area of math was created.

A long time ago, people said that you couldn't divide something like 15÷4; then fractions were invented.

For the Pythagoreans, all numbers were believed to be rational (fractions). The discovery of irrational numbers (numbers that could not be represented as perfect ratios, or fractions) was quite disturbing.

Even as late as Descartes' lifetime (early 1600's) people did not believe that an equation like $x + 7 = 5$ had a solution. Today, negative numbers are taught in middle school.

In every case, it takes a while for a new type of number to be accepted. In fact, the next type of number was so bizarre that they were labeled "imaginary". So now it is time for you to accept that equations like $x^2 + 5 = 0$ actually do have solutions, and that the square root of a negative number is permissible. These imaginary numbers (and don't let the name fool you!) are accepted by mathematicians today.

The Basis for Imaginary Numbers

We will now assume that:

$$\boldsymbol{i} = \sqrt{-1} \quad \underline{\text{or}} \quad i^2 = -1$$

and therefore $\sqrt{-25} = 5i$, etc.

A word of caution:
Only use $\sqrt{c} \cdot \sqrt{d} = \sqrt{c \cdot d}$
if c and d are positive.

Give each value.

1) i^2
2) i^3
3) i^4
4) i^5
5) i^6
6) i^7
7) i^8
8) i^9
9) i^{17}
10) i^{38}
11) i^{-1}
12) i^{-6}

13) $(2i)(3i)$
14) $2i(3 + i)$
15) $\sqrt{-9} \cdot \sqrt{-4}$
16) $(6i)^2$
17) $(10i)^3$
18) $(2i)^4$
19) $(4 + i)(3 + i)$
20) $(6 - i)(5 + i)$
21) $(5 + 2i)(7 + 3i)$
22) $(5 + 2i)(5 - 2i)$
23) $(5 + 2i) + (7 + 3i)$
24) $(5 + 2i) - (7 + 3i)$
25) $(3 + i)^2$
26) $\sqrt[8]{1}$
27) $\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i\right)^2$
28) $\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i\right)^8$
29) What can be concluded from #26 and #28?

Solve for x over the set of *complex numbers*. Try checking your answers to show that it works!

Note: A complex number can be *real* (5), *imaginary* (7*i*), or a combination of both (3 + 8*i*).

30) $x^2 - 9 = 0$
31) $x^2 + 9 = 0$ (Where have we seen this before?)
32) $x^2 + 19 = 0$
33) $x^2 + 10x + 34 = 0$