# $10^{th}$ Grade Assignment – Week #5

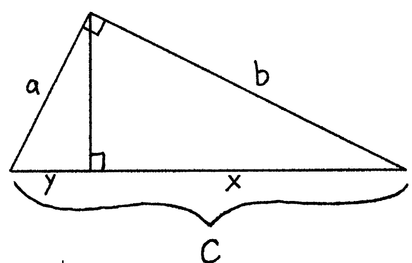
## Individual Work

- **Create your own main lesson book pages**.
  Here are some ideas:
  - "Six proofs of the Pythagorean Theorem".
    With each of the six proofs you have seen, write an explanation (with a drawing) that explains to the reader how the proof "works".
  - "Proofs" essay. Write an essay about the importance of proofs, which may address such ideas/questions as: The Greek search for truth; Why do we need to prove anything? What is a proof? What is a good proof? A perfect proof? The best-possible proof?
  - "How the Ancient Greeks changed the World." Focus on how their way of thinking was a major shift, and how it set the stage for changes to come centuries later.
- *Challenge!* Create a 17-gon by following the (theoretically perfect) instructions given on the below page.
- **Extra Lecture**. If you have the desire and time, continue trying to solve some of the *Euclidean Construction Puzzles* from last week's group assignment. Then, when you are ready, you can watch Lecture #3 (for this week). In this lecture, I show solutions for the eight (possible) Euclidean construction puzzles that I gave you last week - puzzles like constructing the 15-gon and finding the center of a circle. I'm sure you will find it quite interesting. At the end of the lecture, I give two new geometry puzzles, for those who would like the extra challenge.

## Group Assignment:

*for Tuesday:*

- **Proof #6**
  Instructions: Using the diagram shown on the left, explain each of the below steps to each other so that everyone fully understands. Remember that the goal of every proof is to *understand* the reasoning of each step, so that you can then say to yourself: "I know why this theorem is true!"




1) $\frac{a}{y} = \frac{c}{a}$
2) $a^2 = cy$
3) $\frac{b}{x} = \frac{c}{b}$
4) $b^2 = cx$
5) $a^2 + b^2 = cy + cx$
6) $a^2 + b^2 = c(y + x)$
7) $\mathbf{a^2 + b^2 = c^2}$

- **Group Discussion.** Discuss each of the following:
  1) In your opinion, which of the six proofs is the best one? What makes it the best one?
  2) Each of these proofs has at least one (perhaps hidden) assumption. For each of the six proofs, state what it is assuming.
  3) In general, what makes for a good proof?
  4) What would make a proof perfect?
  5) Of the six proofs, which one would Euclid have liked?
  6) On a different note, how do you think you could prove that vertical angles are equal?

# <u>Group Assignment</u>:

*for Thursday:*

- **<u>Euclid's First "Book"</u>**
  The *Summary of Book I of Euclid's Elements* is found at the end of this document.
  Your task is to look it over, and make as much sense of it as you can. **<u>Don't spend more than 30 minutes on this!</u>** Follow these instructions:
  - Here is some helpful terminology:
    - <u>Subtend</u> means "inside" or "in the middle of".
    - An <u>included angle</u> is the same as an "in between" angle.
    - An <u>exterior angle</u> of a triangle is the angle outside the triangle.
    - The <u>converse</u> is the "reverse" of a theorem or statement.
      For example, the statement: "If it is a dog, then it has four legs" has this converse: "If it has four legs, then it is a dog." Note that if a statement or theorem is true, it does not necessary mean that the converse will be true. It has to be proven.
  - Now look at the *Summary of Book I,* which consists of 48 theorems, and do this:
    (Some answers are below, but don't look until after you have seriously thought about it!)
    - Give the theorem number of each of these theorems (which you have seen before, and are listed in order of appearance):
      - SAS Δ ≈ Theorem
      - Isosceles Δ Theorem
      - SSS Δ ≈ Theorem
      - Supplementary Angle Theorem
      - Vertical Angle Theorem
      - ASA Δ ≈ Theorem
      - Alternate Interior, Corresponding, and Same-Side Interior Angle Theorems
      - Angle in a Triangle add to 180°
      - Shear and Stretch theorems
      - Pythagorean Theorem
    - You aren't responsible for understanding #2, 7, 21, 39, 40 – these theorems are used primarily as stepping stones to prove other, more important theorems.
    - If you have the time…
      Find the theorems which are <u>converses</u> of other theorems (always the one listed previously). There are seven converse theorems in this "Book I".
    - For the rest of the theorems, label them as one of the following:
      * meaning "I've seen this before."
      # meaning "This is new, but I understand it."
      ? meaning "I don't understand this one."

- <u>Group Discussion</u>. One can say that: "In Greece, Pythagoras gained intelligence; in Egypt he gained wisdom." What is the difference between intelligence and wisdom?

# Construction of a 17-gon[1]

1. Construct a circle with diameter AB and center C.
2. Construct a radius, perpendicular to AB, with point D on that radius such that $\overline{CD} = \frac{1}{4}\,\overline{CA}$.
3. Find point E on BC such that $\angle EDC = \frac{1}{4}\angle BDC$.
4. Find point H on AC such that $\angle EDH$ is half a right angle (45°).
5. Draw a circle with BH as diameter, labeling its center as X and its intersection with line CD extended as K.
6. Draw a circle with E as its center and EK as its radius. Label this circle's intersection with BC as Y, and its intersection with AC as Z. (Note: X and Y are close, but do not actually coincide.)
7. Draw a line perpendicular to AB from Y and label where it crosses the original circle as $P_4$.
   Draw a line perpendicular to AB from Z and label where it crosses the original circle as $P_6$ (on the same side of AB as $P_4$).
8. Points $P_4$ and $P_6$ are the 4th and 6th vertices of the desired 17-gon. Find $P_5$ by bisecting $\angle P_4CP_6$.
9. Find the remaining vertices of the 17-gon by marking off the distance $\overline{P_4P_5}$ around the perimeter of the circle.
   If done perfectly, point B will be the first vertex ($P_1$), and the radius CA bisects a side of the 17-gon.

[1] This particular construction is by H.W. Richmond (*Mathematische Annalen*, volume 67, 1909).
In 1796, Carl Friedrich Gauss proved that the 17-gon was constructible, but he did not provide a method for doing it. In 1800, Johannes Erchinger provided the first method for constructing the 17-gon.

# The Foundation of Euclid's Elements

## The 34 Definitions (some of the names of the terms have been changed from Heath's original translation.)

1. A **point** is that which has no part.
2. A **curve** is breadthless length.
3. The **extremities of a curve** are points.
4. A **line** is a curve which lies evenly with the points on themselves.
5. A **surface** is that which has length and breadth only.
6. The **extremities of a surface** are curves.
7. A **plane** is a surface which lies evenly with the lines on themselves.
8. A plane **angle** is the inclination to one another of two curves in a plane which meet one another and do not lie in a straight line.
9. When the sides of an angle are straight lines, then the angle is called **rectilineal**.
10. When a line stood up on another line makes the adjacent angles equal to one another, each of the angles is **right**, and the line standing on the other is called a **perpendicular** to that on which it stands.
11. An **obtuse angle** is an angle greater than a right angle.
12. An **acute angle** is an angle less than a right angle.
13. A **boundary** is that which is an extremity of anything.
14. A **figure** is that which is contained by any boundary or boundaries.
15. A **circle** is a plane figure contained by one curve such that all the lines falling upon it from one point among those lying within the figure are equal to one another.
16. And the point is called the **center** of the circle.
17. A **diameter** of the circle is any line drawn through the center and terminated in both directions by the circumference of the circle, and such a line also bisects the circle.
18. A **semi-circle** is the figure contained by the diameter and the circumference cut-off by it. The center of the semi-circle is the same as that of the circle.
19. **Rectilineal figures** are those which are contained by lines. **Triangles** are those contained by three lines, **quadrilaterals** are those contained by four lines, and **multilaterals** are those contained by more than four lines.
20. Of triangles, an **equilateral triangle** has three equal sides; an **isosceles triangle** has two of its sides alone equal; a **scalene** triangle has its three sides unequal, and…
21. A **right triangle** has a right angle; an **obtuse triangle** has an obtuse angle; and an **acute triangle** has its three angles acute.
22. Of quadrilaterals, a **square** is both equilateral and right-angled; an **oblong** is right-angled but not equilateral [i.e., a rectangle which is not a square]; a **rhombus** is equilateral but not right-angled; and a **rhomboid** has its opposite sides and angles equal to one another, but is neither equilateral nor right-angled [i.e., a parallelo-gram which is neither a rhombus nor a rectangle]. Let all other quadrilaterals be called **trapezia.**
23. **Parallel** lines are lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

## The 5 Common Notions

1. Transitive Property. Things which are equal to the same thing are also equal to each other.
2. Addition Property. If equals are added to equals, then the sums are equal.
3. Subtraction Property. If equals are subtracted from equals, then the differences are equal.
4. Things (i.e. figures or solids) that coincide with one another are congruent.
5. The whole is greater than the part.

## The 5 Postulates

1. A line can be drawn between any two given points.
2. Any line can be extended.
3. A circle can be drawn with any center and any distance [as its radius].
4. All right angles are equal.
5. The Parallel Postulate. If a line falling on two lines makes the interior angles on the same side less than two right angles, the two lines, if produced indefinitely, meet on that side on which the angles are less than two right angles.

# A Summary of
# The 13 Books of Euclid's Elements

## Book I Basic constructions and theorems involving angles, triangles, parallelograms, etc.

***Th. 1:*** To construct an equilateral triangle.
***Th. 4:*** SAS triangle congruency theorem.
***Th. 5:*** Isosceles Triangle Theorem.
***Th. 8:*** SSS triangle congruency theorem.
***Th. 13:*** Supplementary Angle Theorem.
***Th. 15:*** Vertical Angle Theorem.
***Th. 26:*** AAS, AAS triangle congruency.
***Th. 29:*** (a) Corresponding angles are equal.
(b) Alternate interior angles are equal.
(c) Same-side interior angles are equal.
***Th. 32:*** The angles in a triangle add to 180°.
***Th. 35-38, 41:*** Shear and Stretch theorems.
***Th. 46:*** To construct a square given one side.
***Th. 47:*** Pythagorean Theorem.

## Book II "Geometrical Algebra" and theorems regarding equal area.

***Th. 4:*** "*If a straight line be cut at random, the square on the whole is equal to the squares on the segments and twice the rectangle contained by the segments.*" This is the geometric equivalent of the identity $(a+b)^2 = a^2 + 2ab + b^2$.
***Th. 5:*** Geometric equivalent of the identity $(a+b)(a-b) = a^2 - b^2$, and used as solution to $ax - x^2 = b^2$.
***Th. 11:*** "*How to cut a given straight line so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining side.*" This is the equivalent of solving $ax + x^2 = a^2$.
***Th. 12, 13:*** Variation of the Pythagorean Theorem for acute and obtuse triangles. (Law of Cosines)
***Th. 14:*** Construction of a square with an area equal to a given polygon.

## Book III Circle geometry

***Th. 1:*** Finding the center of a given circle.
***Th. 17:*** Constructing a tangent line through a given point on a circle.
***Th. 20:*** Inscribed Angle Theorem.
***Th. 21:*** "*Inscribed angles that are subtended by the same arc are equal to one another.*"
***Th. 22:*** Inscribed Quadrilateral Theorem.
***Th. 31:*** Theorem of Thales.
***Th. 32:*** Chord-Tangent Theorem.
***Th. 35:*** Chord-Segment Theorem.
***Th. 36:*** Secant-Tangent Theorem.
***Th. 37:*** Secant-Tangent Theorem converse

## Book IV Constructions of polygons

***Th. 1:*** To draw a chord inside a given circle equal to a given line.
***Th. 2,3:*** To inscribe or circumscribe in or about a given circle a triangle similar to a given triangle.
***Th. 4,5:*** To inscribe or circumscribe a circle in or about a given triangle.
***Th. 6-9:*** To inscribe or circumscribe a square in or about a given circle, and vice versa.
***Th. 10:*** To inscribe a *golden triangle* (the triangle found in a regular pentagon) in a given circle.
***Th. 11-14:*** To inscribe or circumscribe a regular pentagon in or about a given circle, and vice versa.
***Th. 15:*** To inscribe a regular hexagon in a given circle.
***Th. 16:*** To inscribe a regular 15-gon in a given circle.

## Book V The "algebra" of proportions independent of geometry. (Eudoxus' theory of proportion)

***Th. 1:*** States $ma+mb+mc+... = m(a+b+c...)$.
***Th. 4:*** States that if $a:b = c:d$ then $ma:nb = mc:nd$.
***Th. 5:*** States $ma-mb = m(a-b)$.
***Th. 9:*** States that if $a:c = b:c$ then $a=b$.
***Th. 10:*** States that if $a:c > b:c$ then $a>b$.
***Th. 11:*** States that if $a:b = c:d$ and $c:d = e:f$ then $a:b = e:f$.
***Th. 12:*** States that $a:a' = b:b' = c:c'.... = (a+b):(a'+b') = (a+b+c...):(a'+b'+c'...)$.
***Th. 15:*** States $a:b = ma:mb$.
***Th. 16:*** States that if $a:b = c:d$ then $a:c = b:d$.
***Th. 17:*** States that if $a:b = c:d$ then $(a-b):b = (c-d):d$.
***Th. 18:*** States that if $a:b = c:d$ then $(a+b):b = (c+d):d$ [also $a:(a+b) = c:(c+d)$].
***Th. 19:*** States that if $a:b = c:d$ then $a:b = (a-c):(b-d)$.
***Th. 25:*** States that if $a:b = c:d$ and $a$ is the greatest and $d$ is the least, then $a+d > b+c$

## Book VI Proportions relating to geometry.

***Th. 2:*** Triangle proportionality theorem.
***Th. 3:*** Triangle angle-bisector theorem.
***Th. 4:*** Similar triangles have their sides in equal proportions.
***Th. 5:*** SSS similarity theorem.
***Th. 6:*** SAS similarity theorem.
***Th. 7:*** HL similarity theorem.
***Th. 8:*** Altitude of the hypotenuse theorem.
***Th. 11:*** To construct the third term in a geometric progression. (Given x and y, find z such that x:y = y:z).
***Th. 12:*** Given a, b, c to construct x such that a:b = c:x.
***Th. 13:*** To construct the geometric mean. (Given x, z find y such that x:y = y:z).
***Th. 16:*** "*If four lines are proportional, then the rectangle contained by the extremes is equal to the rectangle contained by the means [and vice versa].*" Algebraically, this is: if a:b = c:d then a•d = b•c.
***Th. 25:*** To construct a polygon similar to one polygon and equal in area to another polygon.
***Th. 28, 29:*** Geometric solution to a general quadratic equation.
***Th. 30:*** To cut a given line into the *Golden Ratio* (golden section).
***Th. 31:*** Generalization of the Pythagorean Theorem. If a right triangle has similar figures drawn in the same orientation off each of its three sides, then the area of the largest figure equals the sum of the areas of the other two figures.
***Th. 33:*** With two equal circles, angles have the same ratio as the arcs that subtend them, whether they stand at the centers or at the circumferences.

## Book VII, VIII, IX Elementary number theory.

***Th. VII-2:*** To find the greatest common factor of two numbers.
***Th. VII-16:*** Commutative property of multiplication a•b = b•a
***Th. VII-32:*** Any number is either prime or has some prime number as its factor.
***Th. VII-34:*** To find the least common multiple of two numbers.
***Th. VIII-14:*** If $a^2$ is a factor of $b^2$, then a is a factor of b, and vice versa.
***Th. VIII-16:*** If $a^2$ is *not* a factor of $b^2$, then a is *not* a factor of b, and vice versa.
***Th. VIII-24:*** If $a:b = c^2:d^2$ and c is a square, then d is a square.
***Th. VIII-25:*** If $a:b = c^3:d^3$ and c is a cube, then d is a cube.
***Th. IX-14:*** *The Fundamental Theorem of Arithmetic*, which says that any integer greater than one can be expressed as the product of primes in only one way. (e.g. $24 = 2^3 \cdot 3$)
***Th. IX-20:*** States that the number of primes is infinite.
***Th. IX-35:*** The equivalent of a formula for the sum of the numbers in a geometric progression.
***Th. IX-36:*** *Euclid's Perfect Number Theorem.* If $2^n - 1$ is prime, then $2^{n-1}(2^n - 1)$ is perfect.

## Book X On Irrational Numbers (115 theorems!)

***Th. X-3:*** Given two commensurable magnitudes, to find their common measure (common multiple).
***Th. X-28:*** A method for determining Pythagorean triples.
***Th. X-42:*** The equivalent of saying if $a+\sqrt{b} = x+\sqrt{y}$ then a=x and b=y.
***Th. X-112:*** The equivalent of rationalizing the denominator of a fraction – e.g. $\frac{3}{\sqrt{7}+\sqrt{5}}$ .

## Book XI. XII, XIII Solid geometry.

***Th. XII-2:*** *(The Basis for the Method of Exhaustion)* The areas of two circles have the same ratio as the areas of the squares of their diameters.
***Th. XII-7,10:*** The formula for the volume of a *pyramid* and a *cone*.
***Th. XIII-10:*** A triangle whose sides are respectively the sides of an equilateral pentagon, hexagon, and decagon, all inscribed in the same circle, is a right triangle.
***Th. XIII-13,14,15,16,17:*** The ratio of the edges of a tetrahedron to an octahedron to a cube to an icosahedron to a dodecahedron to the diameter of the sphere that circumscribes them all

is: $$T : O : C : I : D : S = \sqrt{2/3} : \sqrt{1/2} : \sqrt{1/3} : \sqrt{\frac{5-\sqrt{5}}{10}} : \sqrt{\frac{\sqrt{5}-1}{2\sqrt{3}}} : 1$$

***Th. XIII-18:*** There are only five possible regular [Platonic] solids.

# A Summary of **Book I** of Euclid's Elements

**I-1** *Construction* of an equilateral triangle, given one side.

**I-2** *Construction:* To place at a given point (as an extremity) a line equal to a given line.

**I-3** *Construction:* Given two unequal lines, to cut off from the greater line a piece equal to the lesser line.

**I-4** If the two sides and the included angle of one triangle are congruent to the two sides and the included angle of another triangle, then the third sides are equal, the remaining angles are equal, and the two triangles are congruent.

**I-5** In an Isosceles triangle, (a) the base angles are equal to one another, and
(b) if the two sides are extended, then the angles under the bases will be equal to one another.

**I-6** If the two base angles of a given triangle are equal, then the sides are also equal.

**I-7** Given two lines, drawn from the ends of a third line, and meeting together in a point, there cannot be constructed, on the same third line and on the same side of it, two other lines, with the same lengths and placed on the same extremities as the first two lines, such that they meet at a different point.

**I-8** If the three sides of a given triangle are equal, respectively, to the three sides of a second triangle, then all of the angles will be equal, respectively, to one another.

**I-9** *Construction:* Bisection of an angle.

**I-10** *Construction:* Bisection of a line.

**I-11** *Construction* of a line perpendicular to a given line from a point on that line.

**I-12** *Construction* of a line perpendicular to a given line from a point NOT on that line.

**I-13** If two adjacent angles form a straight line, then the sum of the angles are equal to two right angles.

**I-14** If two adjacent angles add to two right angles, then a straight line is formed.

**I-15** Vertical angles are equal.

**I-16** In any triangle, the exterior angle is greater than either of the opposite interior angles.

**I-17** In any triangle, the sum of any two angles is less than two right angles.

**I-18** In any triangle, the greater side subtends the greater angle.

**I-19** In any triangle, the greater angle is subtended by the greater side.

**I-20** In any triangle, any two sides added together are greater than the remaining side.

**I-21** If two lines are drawn from the ends of the base of a triangle, such that the point of intersection of the two lines is within the triangle, then the sum of the two lines will be less than the sum of the two sides of the original triangle, and it will contain a greater angle.

**I-22** *Construction* of a triangle given three lines.

**I-23** *Construction:* Copying a given angle onto a given line.

**I-24** If the two sides of one triangle are equal to two sides of a second triangle, then the triangle with the greater included angle has the greater remaining side.

**I-25** If the two sides of one triangle are equal to the two sides of a second triangle, then the triangle with the greater remaining side has the greater included angle.

**I-26** If two angles and a side of one triangle are equal to two angles and a side of a second triangle, then the triangles are congruent.

**I-27** If two lines are cut by a transversal, and alternate interior angles are equal, then the lines are parallel.

**I-28** If two lines are cut by a transversal, and corresponding angles are equal or the same-side interior angles add to two right angles, then the lines are parallel.

**I-29** If two parallel lines are cut by a transversal, then:
- (a) The alternate interior angles are equal.
- (b) The corresponding angles are equal.
- (c) The same-side interior angles add to two right angles.

**I-30** Two lines that are parallel to the same line are parallel to each other.

**I-31** *Construction* of a line parallel to a given line, through a point not on that line.

**I-32** In any triangle:
- (a) Any exterior angle is equal to the sum of the two opposite interior angles.
- (b) The three interior angles add to two right angles.

**I-33** Two lines that join two equal, parallel lines, are themselves parallel and equal.

**I-34** In a parallelogram:
- (a) Opposite sides and angles are equal.
- (b) A diagonal divides the parallelogram into two congruent triangles.

**I-35** Parallelograms lying on the same base and between the same two parallel lines have equal area.

**I-36** Parallelograms having the same length base and lying between the same two parallel lines have equal area.

**I-37** Triangles lying on the same base and lying between the same two parallel lines have equal area.

**I-38** Triangles having the same length base and lying between the same two parallel lines have equal area.

**I-39** Two congruent triangles sharing the same base, and lying on the same side of that base, [but perhaps a mirror image of each other] lie between the same two parallel lines.

**I-40** Two congruent triangles having the same length base, and lying on the same side of the same line, lie between the same two parallel lines.

**I-41** If a triangle and parallelogram have the same length base and lie between the same two parallel lines, then the area of the parallelogram is twice that of the triangle.

**I-42** *Construction* of a parallelogram given an angle and with an area equal to a given triangle.

**I-43** If, in any parallelogram, two lines are drawn parallel to two adjacent sides of the parallelogram, such that they intersect on a diagonal of the parallelogram, then the two smaller parallelograms formed on opposite sides of the diagonal have equal area.

**I-44** *Construction* of a parallelogram given an angle, a side, and with an area equal to a given triangle.

**I-45** *Construction* of a parallelogram given an angle and with an area equal to a given polygon.

**I-46** *Construction* of a square with a given side.

**I-47** In right triangles, [the area of] the square on the side subtending the right angle is equal to [the sum of the areas of] the squares on the other two sides.

**I-48** In a triangle, if [the area of] the square on one of the sides is equal to [the sum of the areas of] the squares on the remaining two sides of the triangle, then the angle contained by the remaining two sides of the triangle is a right angle.