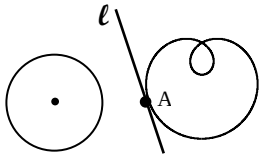
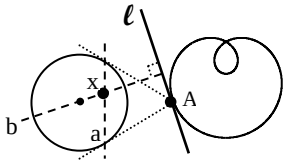
Sequence of Events for Teaching the Polarity of a Curve:

- Go over details of "Tips for Constructing the Polarity of a Curve" (see below).
- It is best to carefully plan which drawing to give to each student. I like to do sequences with particular curves so that once the drawings are complete we can see the curve in movement. If you do this, it is best to use the polarity templates from my website since I have ensured that everything falls on the page nicely.
- Step-by-Step Process for Completing a Drawing for the Polarity of a Curve. Write the following steps on the board:
    1. Finding where the polar curve passes through infinity:
        a. *Lightly* draw all tangents lines to the curve that pass through the center of the circle.
        b. The polar of each of these tangent lines is a point at infinity in the direction that is perpendicular to the tangent line. In each case, show an arrow (with a long tail) at the edge of the page (lightly in pencil) that indicates the direction that the polar curve goes to infinity. Show it also in the opposite direction on the other side of the page. We will number these arrows later.
        c. With each of these tangent lines (that pass through the center of the circle) mark the points of tangency on the original curve.
        d. Erase all tangent lines.
    2. Mark around 20 points on the curve (some of which come from the above special tangent lines). Carefully think about where the points need to be closer together. Label them in order.
    3. Label the arrows (from step #1b) with its proper number (which should corresponds to the point of the curve that had a tangent line passing through the center of the circle).
    4. Finding the Polar Points:
        a. Choose one point on the curve and *carefully* draw (lightly in pencil) a tangent line through that point.
        b. Find the polar point, and mark it in ink. Label it with the correct number. Ask yourself if it makes sense!
        c. Erase the tangent line, all constructions lines, and any tick marks.
        d. Repeat the process for all points on the original curve until all corresponding points of the polar curve have been located. It may become apparent that more points are needed.
    5. Lightly in pencil, connect the points of the polar curve. Pay special attention to the direction(s) that the curve goes to infinity (given by the arrows in step #1). Ask the teacher to check your work.
    6. Color the curve nicely with the required colors.
    7. *Very neatly*, add the title (2.5cm height) and your name (1.5cm height). Make sure the page is in the proper orientation – up is up!

Tips for Constructing the Polarity of a Curve (Taken from my High School Source Book)

- Instead of thinking of the curve as a collection of points, or, alternatively, as a collection of (tangent) lines, we should think of it as a collection of "locations", where each location on the curve has both a point and a tangent line through that point.
- For most curves, choosing between 25 and 30 points on the original curve should be sufficient. These points should not be equally spread out; the points should be closest together wherever the tangent lines are turning the fastest, and wherever there is a cusp or a point of inflexion. The points should be labeled in numerical order as you travel around the curve. Through each point, tangent lines must be drawn as accurately as possible. Be sure to include tangent lines that pass through the center of the polarity conic, as well as tangent lines that are tangent to both the curve and the polarity conic.
- If we are taking the polar of a tangent line (ℓ) that falls outside the polarity conic, then we need to take the polar of any two points on the line. It is best to have the two points be the point of tangency (A), and the tangent line's infinitely distant point (B). The process is conceptually challenging, but easy to execute. First we take the polar of point A, which is line a (which must cross through the polarity conic). Now we will take the polar of point B (the infinitely distant point on line ℓ). The key is to realize that the polar of a point at infinity is a line that runs through the center of the polarity conic and is *perpendicular to the direction of the point.* Therefore, to get line b, we just draw a perpendicular line to line ℓ, such that it passes through the center of the circle. Now we have found our goal: the polar point to line ℓ; it is point x – the point of intersection of lines a and b. By using a drawing triangle, this can be done very quickly.

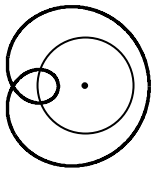




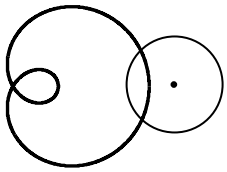
- If we only took the polars of the points of the original curve, then our result would be a collection of tangent lines, and we likely wouldn't be able to tell what the curve really looked like (unless it was a conic). Two very helpful tips: (1) <u>It will simplify matters greatly to only take the polars of the original curve's tangent lines</u>, which will give us the desired points on the polar curve; (2) Draw one tangent line at a time on the original curve, find the polar point, then erase all construction lines.
- Pay special attention to where the original curve's tangent lines pass through the center of the polarity conic. This tells where (the points of) the polar curve must pass through infinity. *If the original curve's tangent line passes through the center of the polarity conic, then the polar curve must pass through infinity in a direction that is perpendicular to that tangent line.*
- By just looking at the original curve and the polarity conic, it is impossible to tell where the resulting curve will fall on the page. Therefore it is best to use my polarity templates (on the following pages or on my website) in order to ensure that the results fall nicely on the page.

Key questions to think about:

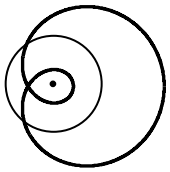
- By looking at the original curve and the polarity conic, how can we tell how many times the resulting curve will pass through infinity? Answer: It depends on the number of times the original curve has tangent lines passing through the center of the polarity conic. To show this clearly, draw three limaçons on the board, as shown below.



The polar curve passes through infinity twice.



The polar curve passes through infinity four times.



The polar curve doesn't pass through infinity at all.

- If the original curve has a cusp, what is happening here on the polar curve? Answer: You have a point of inflection.
- If the original curve has a point of inflection, what is happening here on the polar curve? Answer: You have a cusp.
- If the original curve has a point where the curve passes through twice, what is happening here on the polar curve? Answer: The polar curve has a tangent line with two points on it.
- If the center of the polarity conic lies on the original curve, what is happening here on the polar curve? Answer: The polar curve must be tangent to the line at infinity, like a parabola.