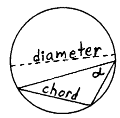
Proofs of the Three Trigonometric Laws

Law of Sines

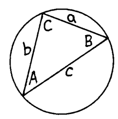
<u>Note</u>: This appears as an exercise in the 10^{th} grade workbook, in the unit *Intro to Trigonometry*.

1. An alternative definition of *Sine* involves an angle inscribed in a circle, and the subtended chord. $sin(\alpha)$ then answers the question: "*The chord is what proportion of the diameter*?"



   It is then written as $sin(\alpha) = \frac{\text{chord}}{\text{diameter}}$

2. Given any ΔABC, we can circumscribe a circle around it.



3. Let d be the diameter of the circle. It follows then that:

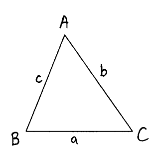
$$\text{Sin}(A) = \frac{a}{d} \quad \text{and} \quad \text{Sin}(B) = \frac{b}{d}$$

Therefore $d = \frac{a}{\sin(A)} = \frac{b}{\sin(B)}$ and $\boxed{\frac{a}{b} = \frac{\sin A}{\sin B}}$

Law of Cosines

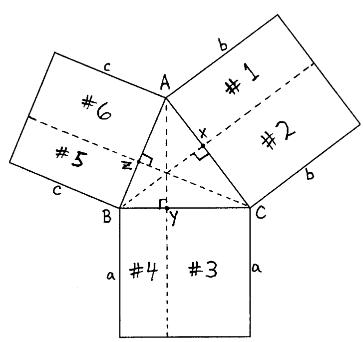
<u>Note</u>: This appears as an exercise in the 11^{th} grade workbook, in the unit *Trigonometry Part II*.

1. Given acute triangle, ΔABC, attach squares to the sides of the triangle, draw altitudes to the sides of the triangle, and label everything as shown below.



2. Using ΔABX, we see that

   AX = c•cos(A).
   Similarly, we have
   CX = a•cos(C)
   BY = c•cos(B)
   CY = b•cos(C)
   AZ = b•cos(A)
   BZ = a•cos(B)



3. The areas of the six rectangles are:
   rect#3 = rect#2 = ab•cos(C)
   rect#5 = rect#4 = ac•cos(B)
   rect#6 = rect#1 = bc•cos(A)

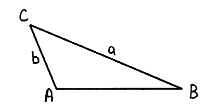
4. The final steps are then:

$$\text{rect\#5} + \text{rect\#6} + \text{rect\#3} = \text{rect\#1} + \text{rect\#2} + \text{rect\#4}$$
$$c^2 + \text{rect\#3} = b^2 + \text{rect\#4}$$
$$c^2 + \text{rect\#3} + \text{rect\#3} = b^2 + \text{rect\#4} + \text{rect\#3}$$
$$c^2 + 2(\text{rect\#3}) = b^2 + a^2$$

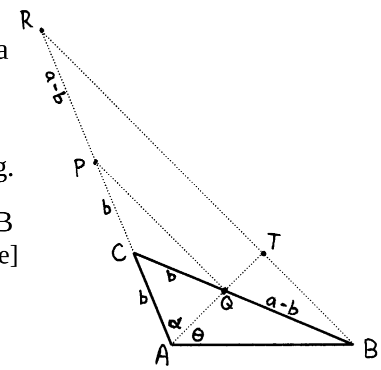
$$\boxed{c^2 = a^2 + b^2 - 2ab\cdot\cos(C)}$$

Law of Tangents

<u>Note</u>: This appears as an exercise in the 11^{th} grade workbook, in the unit *Trigonometry Part II*.



1. Given any triangle, ΔABC, we extend side b upward by the length of side a up to point R. Points P and Q are found by measuring a distance of b away from point C. Lines are then joined as shown in the drawing.



2. PQ is parallel to RB
   [Δ Prop. Th. converse]

The angle measures are as follows:

3. $\angle ACQ = 180° - \angle A - \angle B$
   $\alpha = \angle CQA = \angle TQB$
   $\angle PCQ = \angle A + \angle B = \alpha + \angle CQA = 2\alpha$
   $\boldsymbol{\alpha = \tfrac{1}{2}(\angle A + \angle B)}$

4. $\alpha = \tfrac{1}{2}(\angle A+\angle B) = \angle CQA = \theta + \angle B$
   $\tfrac{1}{2}\angle A + \tfrac{1}{2}\angle B = \theta + \angle B$
   $\boldsymbol{\theta = \tfrac{1}{2}(\angle A - \angle B)}$

5. $\angle PCQ + \angle CPQ + \angle PQC = 180°$
   $2\alpha + 2\angle PQC = 180° \rightarrow \alpha + \angle PQC = 90°$
   $\boldsymbol{\angle AQP = 90° = \angle ATR}$

6. Let x = AT. Using ΔATB and ΔATR, we get TB = x•tan(θ) and TR = x•tan(α)

7. The last few steps are then:
   ΔTQB ~ ΔPQA ~ ΔRTA
   TB:QB = TR:AR
   x•tan(θ) : (a–b) = x•tan(α) : (a+b)
   x•tan(θ) : x•tan(α) = (a–b) : (a+b)

$$\boxed{\frac{\tan[\tfrac{1}{2}(A-B)]}{\tan[\tfrac{1}{2}(A+B)]} = \frac{a-b}{a+b}}$$

where ½(A+B) is the average of ∠A and ∠B.

and ½(A–B) is the difference between this average and either of the two angles.