—— *Possibility and Probability* (9th Grade Main Lesson) ——

Overview

The subject of *possibility and probability* (which includes permutations and combinations) concentrates on answering questions such as: How many possible (shortest) routes are there for going between two points marked on a grid? How many ways are there for twenty students to get in line? How many different committees of three can be formed from a group of ten people? What is the probability of flipping five coins and getting all heads?

These questions often yield surprising results. It is through careful thinking that we can overcome the task of making sense of these difficult problems. We recognize patterns and similarities with previously encountered problems and learn to solve the problems in a systematic way.

The students work both independently and in groups to solve problems. Carefully worded explanations of a few solutions are written in their lesson books. Each student is challenged at some point during the course, and hopefully develops confidence in their thinking.

Lesson plans

Detailed day-by-day lesson plans can be downloaded from our website: **www.JamieYorkPress.com**.

The big picture

This block is only intended to be an introduction to this subject area. These problems can get very difficult. In this block we are working with the big picture. We only do a few central problems very thoroughly. The purpose here is not to work on skills; we don't expect the students to become proficient at these problems during this main lesson block. Later in the year, they will revisit this topic in the track class as they work with the *Possibility & Probability* unit in our *9th Grade Workbook*. This is when the students get to practice solving many of these problems. Then in eleventh grade they revisit the topic once again and work with more challenging problems in the *Possibility & Probability* unit of our *11th/12th Grade Workbook*.

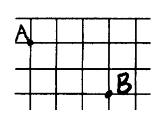
Learning to think independently

At the start of this block, I give a motivational speech about the importance of learning to think for yourself. I make the point that most people today simply believe what they are told (e.g., in the TV news, or on the Internet). I say to the students: "In the lower grades, you were expected to believe what your teacher told you. Your teacher probably told you that π was approximately 3.14. You believed your teacher. It is now time for you to start thinking for yourself." Here is the key thought for our block:

> *"Don't just believe what you're told. Believe it only when you know – in your own thinking – that it's true."*

Central questions

- With each of the below questions, I try to get the students to discover the answer as much as possible on their own; I resist the temptation to give them answers. Students present their ideas to the class, and slowly everyone comes to clarity about the correct solution.
- After they have reached clarity about a solution, they write up the problem in their main lesson books. For each essay, I expect a statement of the question, a description of what they did (even if they made an error), and a full explanation of the correct solution.
- *The Street Problem.* How many shortest routes are there from A to B?



    - I begin the block with this question. The students inevitably draw out all of the possible routes, and usually come up with the correct answer (10).
    - Then I ask how many possibilities would there be for a 5x6 grid? Finding a method to answer this question becomes a goal for the block.
- *The Wardrobe Problem.* How many possible outfits can you choose to wear if you have 3 pants and 5 shirts to choose from?
    - This leads to the *Fundamental Counting Principle* (see the Summary Sheet below).
- *The Seating Chart Problem.* How many possibilities are there for making a seating chart for the class?
    - This leads to the idea of factorial.

- I also ask the following question (which leads to surprising results):
  Question: If you rearranged the class into a different seating chart every second, how long would it take before you exhausted every possibility?
  Solution: For a group of size n, we get: t = n! seconds÷3600sec/hour÷24hr/day÷365days/year
  If n=13, n!=6,227,020,800 $t \approx 197$ years.
  If n=18, n!≈$6.40 \cdot 10^{15}$ $t \approx 203{,}000{,}000$ years
  If n=25, n!≈$1.55 \cdot 10^{25}$ $t \approx 492{,}000{,}000{,}000{,}000{,}000$ years
- *The License Plate Problem*. How many possibilities are there for a license plate that has 3 digits followed by 3 letters? (Answer: $10^3 \cdot 26^3$) This is an extension of the *Fundamental Counting Principle.*
- *The Prize Problem*. How many possibilities are there for giving out 1st, 2nd, 3rd place prizes in a race that has 53 participants?
  - This leads to the idea of a *Permutation* (see the Summary Sheet, below).
  - The notation is ${}_nP_r$, which means the first r numbers of n factorial multiplied together.
- *The Committee Problem*. How many possibilities there are for selecting 3 people out of 5 to be on a committee.
  - First ask: how many possibilities are there for selecting a Pres, VP, and Sec from a group of 5 people? Then ask the above question and make sure they understand the difference.
  - This is probably the most difficult step in the block. We should build up slowly to an understanding of the below table:

Total # of people	# of people selected	# of possibilities for prize problem	# of possibilities for committee problem
5	3	${}_5P_3 = 60$	${}_5C_3 = \frac{{}_5P_3}{3!} = \frac{60}{6} = 10$
7	2	${}_7P_2 = 42$	${}_7C_2 = \frac{{}_7P_2}{2!} = \frac{42}{2} = 21$
10	3	${}_{10}P_3 = 720$	${}_{10}C_3 = \frac{{}_{10}P_3}{3!} = \frac{720}{6} = 120$
20	4	${}_{20}P_4 = 116{,}280$	${}_{20}C_4 = \frac{{}_{20}P_4}{4!} = \frac{116280}{6} = 4845$
75	5	${}_{75}P_5 = 2{,}071{,}126{,}800$	${}_{75}C_5 = \frac{{}_5P_3}{3!} = \frac{60}{6} = 17{,}259{,}390$

  - This leads to the idea of a *Combination* (see the Summary Sheet, below).
  - The notation is ${}_nC_r$, which means ${}_nP_r$ divided by r!
- *Word Scrambling*.
  - Build up to this slowly by asking students to figure how many ways there are to arrange the letters in these words: MARK, TARA, MOMO, JESSE, BOBBY, etc.
  - This leads to the idea of *Distinguishable Arrangements* (see the Summary Sheet, below).
- *Introduction to Probability*.
  - We only give a brief intro to probability in this block. We go into much more detail, and do much more challenging problems, in eleventh grade with the *Possibility & Probability* unit of our *11th/12th Grade Workbook*.
  - Introduce the formula $P = \frac{\text{number of successful outcomes}}{\text{number of total possible outcomes}}$
  - But we must make clear – this formula assumes that each possibility is equally likely.
    Example: When flipping a coin, we have a 50% chance of getting "heads" because each of the two possible outcomes is equally likely. When shooting a basketball, there are two possible outcomes when taking a shot: either you make a basket, or not. However, it is not correct to then say that you have a 50% chance to make a basket. The probability of making it depends upon many factors. Likewise, we can think that there are 11 possible outcomes (2-12) for (the sum of) rolling two dice. However, there isn't a $^1/_{11}$ probability of rolling a 10 because each of the 11 possible outcomes is *not* equally likely.

− Possibility and Probability −

- *Rolling Two Dice*. What is the probability of rolling a (sum of) 10 with two dice?
  - We begin this problem by doing an experiment. Each student rolls a pair of dice 50 times and records the number of occurrences of each value. They then create a spreadsheet (e.g., Excel) to record the results in a table. (This is a good opportunity to do a brief lesson on how to use formulas in a spreadsheet.) If there are 20 students in the class, we will have 1000 results. The compiled results should then be studied, and interesting observations noted.
  - Then we can begin to address the main question: what is the (theoretical) probability of rolling a 10 with two dice? We list all 36 *equally likely* possibilities in order to see that three of them result in a 10. Therefore the probability is $^3/_{36} \approx 8\frac{1}{3}\%$.
  - Now we can fill out the following table:

    Theoretical Probabilities for Rolling Two Dice

total roll	# successes	Probability (%)
2	1	2.78
3	2	5.56
4	3	8.33
5	4	11.11
6	5	13.89
7	6	16.67
8	5	13.89
9	4	11.11
10	3	8.33
11	2	5.56
12	1	2.78

  - Now we can discuss how our experimental data compares with our theoretical answer. What can we say about this? How would it be different if we had rolled only 20 times? Or if we had rolled one million times?
  - This should lead to the *Statistical Probability Law*: *The more an event is repeated, the closer the average outcome gets to the expected (theoretical) outcome.*
- *Pascal's Triangle*. (See the next Summary Sheet for details.)
  - Have students point out various patterns that they notice.
  - Pascal's Triangle can help us solve probability problems.
- *The Street Problem – Revisited*: The 5x6 Street Grid problem.
  - Point out that the Street Grid problem is found within Pascal's Triangle – e.g., the number of possible routes with a 3x2 grid is ${}_5C_3$ (= 10), and with a 6x5 grid it is ${}_{11}C_6$ (= 462).
    - To get to any point (x) in the middle of the grid, you have to come either from the point above x, or the point to the left of x. Therefore the number of possible (shortest) routes to x must be the sum of the number of routes to get to those two neighboring points.
  - With a 6x5 street grid, one route is to travel first 6 blocks east, and then 5 blocks south. This can be represented as EEEEEESSSSS. Therefore each possible route is simply a rearrangement of the string of letters EEEEEESSSSS $\rightarrow \frac{11!}{6!\,5!}$ (= 462).
  - Amazingly, we now see how rearranging letters, the committee problem, the street grid problem, and Pascal's triangle are all related.
- *Birthday Problem*. In a (random) group of 50 people, what is the probability that at least two people have the same birthday?
  - This is the most advanced probability problem that we solve.
  - We ask instead: What is the probability that no two people have a common birthday?
    - If a twentieth person enters a room, the probability that he will not have a common birthday with anyone else in the room (all having different birthdays) is 346/365. This allows us to say that the probability of 50 people having no common birthday is:

      $P = \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdot \frac{362}{365} \cdot \ldots \cdot \frac{316}{365} \approx 0.030$, which means there is only a 3% chance that the 50 people all have different birthdays. Therefore, there is a 97% probability that there is at least one common birthday.

Summary Sheet for
Possibility and Probability

The Fundamental Counting Principle (wardrobe problem)

The combined number of ways for two (independent) things to happen is the product of the number of ways for each of those things to happen separately.

Example: How many different outfits can Fred wear if he has 10 shirts and 8 pairs of pants?
Solution: For each of the 10 shirts he has 8 options for pants. Therefore, $10 \cdot 8 = \underline{\mathbf{80}}$.

Permutations (prize problem)

The number of ways to select r out of n items, and put them into order is:

$${}_nP_r = \frac{n!}{(n-r)!} \quad \text{which is the first r items in n!}$$

Example: How many ways can the top 5 prizes be given out if there are 12 participants?
Solution: ${}_{12}P_5 = \frac{12!}{7!} \rightarrow 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \rightarrow \underline{\mathbf{95{,}040}}$

Combinations (committee problem)

The number of ways to choose a group of r out of n items is:

$${}_nC_r = \frac{{}_nP_r}{r!} = \frac{n!}{(n-r)!\,r!}$$

Example: How many committees of 5 people can be chosen from a group of 12?
Solution: ${}_{12}C_5 = \frac{12!}{7!\,5!} \rightarrow \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \rightarrow \underline{\mathbf{792}}$

Distinguishable Arrangements (rearranging letters)

With a total of n items, of which a items are indistinguishable from each other, as are a further b items, etc., the number of distinguishable arrangements (or permutations) is:

$$\frac{n!}{a!\,b!\,c!\,\ldots}$$

Example: How many ways are there to reorder the letters AAABBCCCCDEEE
Solution: $\frac{13!}{3!\,2!\,4!\,3!} \rightarrow \underline{\mathbf{3{,}603{,}600}}$

The Probability of an Event

The probability of an event successfully occurring, P(E), *is equal to the number of possible (equally likely) success outcomes divided by the total number of possible (equally likely) outcomes.*

$$P(E) = \frac{\text{number of successful outcomes}}{\text{number of total possible outcomes}}$$

Example: One card is drawn from a standard deck. Find the probability of getting a spade.
Solution: The probability is $\frac{13}{52}$ or **<u>25%</u>**.

Sets that are not Mutually Exclusive

If A and B share members, then the number of members that are either in A or B is A plus B minus the number of members in both A and B.

$$A \cup B = A + B - (A \cap B)$$

Example: Everyone in a class is either a sophomore or a girl. 15 are sophomores and 12 are girls. If there are 8 students who are sophomore girls, how many are in the class?
Solution: $15 + 12 - 8 = \underline{\mathbf{19}}$.

Summary Sheet for Possibility and Probability (continued)

Two Independent Events

If A and B are independent events, the probability that both A and B will occur is the product of probabilities of each occurring separately.

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

Example: If you roll a die and flip a coin, what is the probability of getting a 2 and a head?
Solution: $\frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12}$

Statistical Probability

The more an event is repeated, the closer the average outcome gets to the expected (theoretical) outcome.

Example: If we flip a coin n times, we expect that the number of heads will get closer to 50% for larger values of n.

Example: Since the probability of rolling a sum of a 9 with two dice is 11.1%, we expect that if we roll two dice one thousand times, then we will get a 9 about 11% of the time.

The Probability of a Complement

If A is the complement of B, then the sum of their probabilities is equal to one, or 100%.

Example: What is the probability of flipping three coins and getting at least one head?
Solution: The complement (or opposite) of this is getting no heads, which has a probability of $^1/_8$. Therefore, the probability of getting at least one head is $1 - {}^1/_8 = {}^7/_8$.

Pascal's Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1

Some properties of Pascal's Triangle:

- If we start counting with zero, then the number in the n^{th} row and r^{th} column is ${}_nC_r$.
- From the property that was used to generate the triangle, we have ${}_{n-1}C_r + {}_{n-1}C_{r-1} = {}_nC_r$
- The sum of the numbers in the n^{th} row is 2^n. In other words: $\sum_{r=1}^{n} {}_nC_r = 2^n$
- *The Binomial Theorem.* The n^{th} row is the set of coefficients in the expansion of the binomial $(x+y)^n$, or $(x+1)^n$.
- The sum of the numbers in each "shallow" diagonal is a Fibonacci number.