Solution

A Group of Teenagers.

a) 18,726,552 → There are two 13-year-olds, three 18-year-olds, and a 19-year-old.

b) We can tell that there is at least one 13-year-old if the given product is evenly divisible by 13. We can also look at the prime factorization. With the previous problem, the prime factorization of 18,726,552 is $2^3 \cdot 3^6 \cdot 13^2 \cdot 19$. This clearly shows that there are two 13-year-olds and one 19-year-old. 13 and 19 are prime numbers, which makes it easy to determine how many are of that age. Determining the number of 18-year-olds (which isn't a prime number) is more complicated, as we will see in the next problem…

c) Because 18 isn't prime, it is actually possible that the given product is divisible by 18, but there are no 18-year-olds in the group. 18 has prime factors of 2 and 3, and only one other teen number, 15, also has a factor of 3. And because 15 is the only teen number with a prime factor of 5, we first determine how many 15-year-olds are in the group (by dividing by 15), and then use the resulting left-over product to determine the number of 18-year-olds in the group by dividing by 18. Alternatively, we can once again use prime factorization to reach a solution, as is explained below.

d) 17,442,000 → Interestingly, this product is divisible by 18, but there are no 18-year-olds in the group. The prime factorization is $2^4 \cdot 3^3 \cdot 5^3 \cdot 17 \cdot 19$. We easily can see that there is one 17-year-old and one 19-year-old. Because the exponent on the 5 is a 3, we know there must be three 15-year-olds (again, because 15 is the only teen number that has a prime factor of 5). The three 15's will also take all three of the 3's. This means there are no 3's left to allow for any 18-year-olds. The four 2's are all that is left over, which accounts for one 16-year-old.

e) With all of the below problems, we do the prime teen ages (13, 17, 19) first, then we do 15 (factor of 5) and 14 (factor of 7). Then, with whatever is left-over, we figure out the 18's, and lastly, the 16's.

   i) $2^{12} \cdot 13^4 \cdot 17$ → four 13-year-olds, one 17-year-old, and three 16-year-olds.

   ii) $2^{12} \cdot 13^4 \cdot 11$ → impossible because of the 11.

   iii) $2^4 \cdot 3^3 \cdot 5^3 \cdot 7^4 \cdot 17^2$ → two 17-year-olds, four 14-year-olds, three 15-year-olds.

   iv) $2 \cdot 3^4 \cdot 5^6 \cdot 19^3$ → impossible because there must be at least as many 3's as 5's.

   v) $2 \cdot 3^6 \cdot 5^4 \cdot 19^3$ → three 19-year-olds, four 15-year-olds, and one 18-year-old.

   vi) $2^{10} \cdot 3^6 \cdot 7^3 \cdot 13^4 \cdot 17^3$ → three 17-year-olds, four 13-year-olds, three 14-year-olds, three 18-year-olds, and one 16-year-old.

   vii) $2^{19} \cdot 3^8 \cdot 5^2 \cdot 7^4 \cdot 17^2 \cdot 19^3$ → three 19-year-olds, two 17-year-olds, four 14-year-olds, two 15-year-olds, three 18-year-olds, and three 16-year-olds.

   viii) $2^4 \cdot 3^6 \cdot 5^4 \cdot 7^3 \cdot 13$ → one 13-year-old, three 14-year-olds, four 15-year-olds, and one 18-year-old.

   ix) $2^{15} \cdot 3^{10} \cdot 5^2 \cdot 7 \cdot 13^3 \cdot 19^2$ → Impossible. At first, we get two 15-year-olds, which leaves us with eight 3's. Eight 3's mean that we have four 18-year-olds (which requires eight 3's and four 2's). Since the 14-year-old also required one 2, we now have ten 2's remaining. This is not possible because the number of remaining 2's have to be a multiple of four (since $16 = 2^4$). If the original exponent for the 2 was 5 or 9 or 13 or 17, etc., then the problem would have been possible.