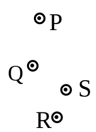
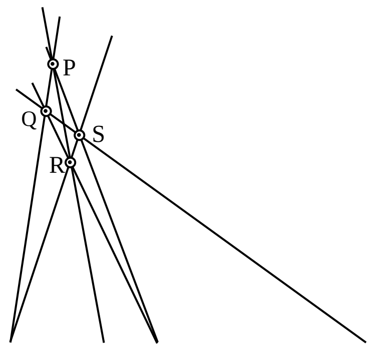
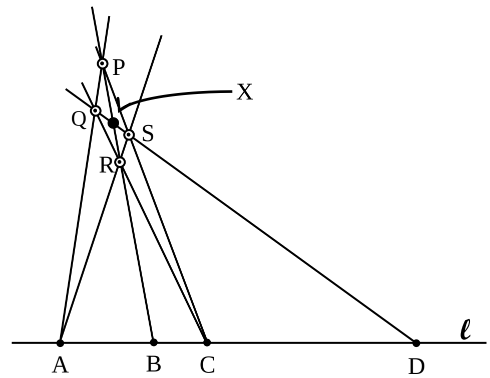
Harmonic Conjugates

- *Quadrangle*

Any four points (no three of which are collinear) define a quadrangle (PQRS, as shown below, left). There are six lines which can be drawn that connect pairs of the quadrangle's points (as shown below, middle). These six points meet at three new points of intersection: A, C, X, which form the quadrangle's *Diagonal Triangle*. Draw a line (**ℓ**) through any two of the three points of the diagonal triangle (in the below drawing, we have chosen points A and C). The six lines of the quadrangle meet line **ℓ** in four points. Points A and C are said to be *harmonic conjugates* of points B and D.



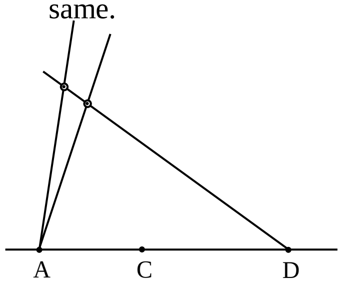


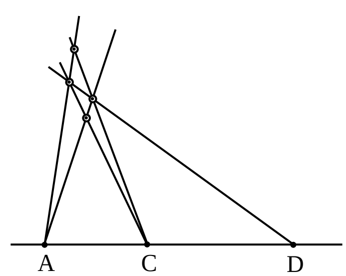


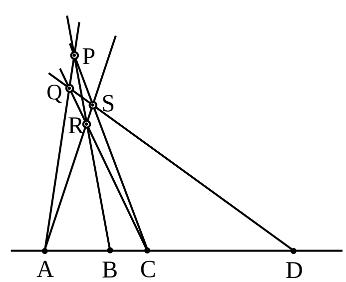
- *Creating harmonic conjugates and a harmonic net*

To create pairs of harmonic conjugates, we essentially work backwards from what was done above. We start with any three points (A, C, D) on a line. Our goal is to locate point B such that points A and C are harmonic conjugates with points B and D. We start by drawing any two lines through point A, and one line through point D. (See below drawings.) This gives us two points of intersection through which we draw two new lines to point C. Two new points of intersection are formed, through which a line is drawn that locates point B.

It is surprising to learn that if we had chosen three different initial lines through points A and D, then we would produce different quadrangles, but the final location of point B would be exactly the same.

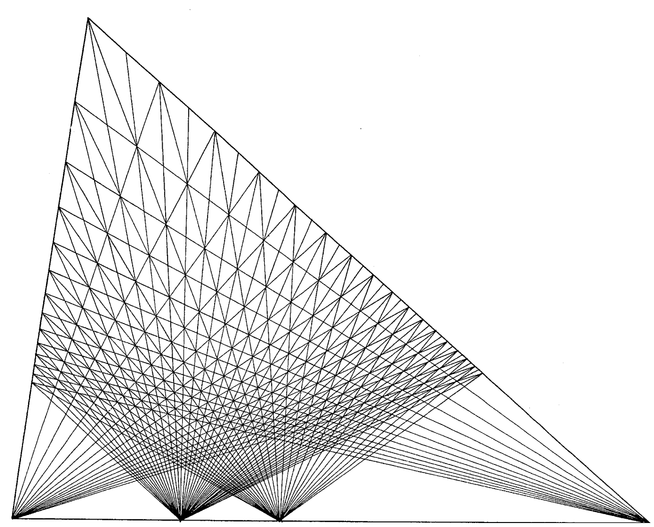






Let PQRS be the original quadrangle, with diagonals passing through B and D. We can now create neighboring quadrangles by drawing new diagonal lines (PD, RD, SB, QB) and then drawing more lines through A and C to complete the new quadrangles. Continuing in this manner creates a *harmonic net*, as shown on the right.

Surprisingly, if you erase all the diagonal lines (that go through points B and D), this harmonic net can be used as an alternative coordinate plane.



A Harmonic Net