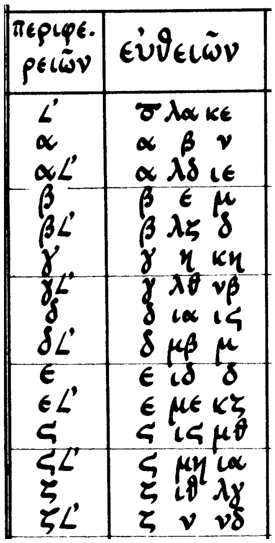
Ptolemy's Trig Tables

(See Trig unit of the 10^{th} Grade workbook for an explanation)

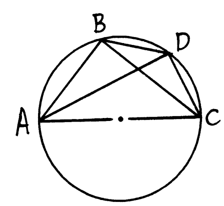


περιφε-ρειῶν	εὐθειῶν
∠'	ō λα κε
α	α β ν
α∠'	α λδ ιε
β	β ε μ
β∠'	β λζ δ
γ	γ η κη
γ∠'	γ λθ νβ
δ	δ ια ις
δ∠'	δ μβ μ
ε	ε ιδ δ
ε∠'	ε με κζ
ς	ς ις μθ
ς∠'	ς μη ια
ζ	ζ ιθ λγ
ζ∠'	ζ ν νδ

Arcs	**Chords**	**Decimal**
1) ½	0, 31,25	0.5236
2) 1	1, 2, 50	1.047
3) 1½	1, 34, 15	1.571
4) 2	2, 5, 40	2.094
5) 2½	2, 37, 4	2.618
6) 3	3, 8, 28	3.141
7) 3½	3, 39, 52	3.664
8) 4	4, 11,16	4.188
9) 4½	4, 42, 40	4.711
10) 5	5, 14, 4	5.234
11) 5½	5, 45, 27	5.758
12) 6	6, 16, 49	6.28
13) 6½	6, 48, 11	6.803
14) 7	7, 19, 33	7.326
15) 7½	7, 50,54	7.8483

Proof of the Half-Angle Formula

- This proof is largely modeled after Ptolemy's proof of his half-angle formula.

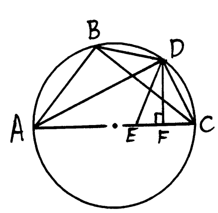


The Purpose of the Proof

- We are given angle α (in the drawing, $\alpha = \angle BAC$), and we start off knowing the value of $sin(\alpha)$ and $cos(\alpha)$. Also, AD bisects α. Our goal is to find the *sine* of half of α. This means that in the drawing we aim to find the *sine* of ∠DAC.

The Proof

1) Place F on AC such that $DF \perp AC$.
   Place E on AC such that $AB \cong AE$.
2) $\Delta BAD \cong \Delta EAD$; $BD \cong DE$; $BD \cong CD$;
   $DE \cong CD$
3) ΔCDE is isosceles; $EF \cong CF$
4) $AC = AE + EF + CF$;
   $AC = AB + CF + CF$;
   $CF = ½(AC - AB)$
5) $CD^2 = CF^2 + DF^2$;
   $DF^2 = AF \cdot CF$;
   $CD^2 = CF^2 + AF \cdot CF$;
   $CD^2 = CF(CF+AF)$;
   $CD^2 = CF \cdot AC$;
   $CD^2 = AC(½(AC - AB)) = ½AC(AC - AB)$

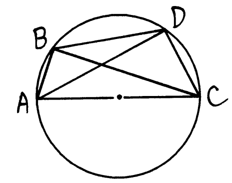


6) $sin(^{\alpha}/_{2}) = \frac{CD}{AC}$; $CD = AC\, sin(^{\alpha}/_{2})$;
   $cos(\alpha) = \frac{AB}{AC}$; $AB = AC\, cos(\alpha)$
7) $[AC\, sin(^{\alpha}/_{2})]^2 = ½AC(AC - AC\, cos(\alpha))$;
   $AC^2\, [sin(^{\alpha}/_{2})]^2 = ½AC^2 - ½AC^2\, cos(\alpha)$;
   $[sin(^{\alpha}/_{2})]^2 = ½ - ½\, cos(\alpha)$;

$$sin(^{\alpha}/_{2}) = \sqrt{½ - ½\, cos(\alpha)}$$

Proof of the Difference Formula

- This proof is largely modeled after Ptolemy's proof of his difference formula.



The Purpose of the Proof

- We are given two angles: α and β (in the drawing $\alpha = \angle BAC$ and $\beta = \angle DAC$). We also know the values of both the sine and cosine of these two angles. Our goal is to find $sin(\alpha - \beta)$. This means that in the drawing we aim to find the *sine* of $\angle BAD$.

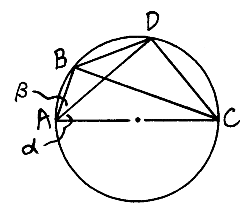
The Proof

1) From Ptolemy's quadrilateral theorem we get $AC \cdot BD + AB \cdot CD = AD \cdot BC$
which then becomes $AC \cdot BD = AD \cdot BC - AB \cdot CD$
2) $cos(\beta) = \frac{AD}{AC} \rightarrow AD = AC\, cos(\beta)$; Likewise, we get $BC = AC\, sin(\alpha)$; $AB = AC\, cos(\alpha)$;
$CD = AC\, sin(\beta)$; $BD = AC\, sin(\alpha - \beta)$;
3) Substituting the values from #2 into the equation from #1 gives us:
$AC[AC\, sin(\alpha - \beta)] = [AC\, cos(\beta)] \cdot [AC\, sin(\alpha)] - [AC\, cos(\alpha)] \cdot [AC\, sin(\beta)]$
Dividing both sides by AC^2 we get:

$$\boxed{sin(\alpha - \beta) = cos(\beta) \cdot sin(\alpha) - cos(\alpha) \cdot sin(\beta)}$$

Proof of the Sum Formula

- This proof is largely modeled after Ptolemy's proof of his sum formula.

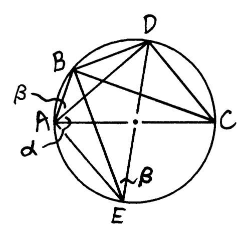


The Purpose of the Proof

- We are given two angles: α and β. We also know the values of both the sine and cosine of these two angles. Our goal is to find $cos(\alpha + \beta)$. This means that in the drawing we aim to find the *cosine* of $\angle BAC$.

The Proof

1) We begin by placing point E on the circle such that $BE \perp BD$, and then drawing DE and AE.
2) Using Ptolemy's quadrilateral theorem with ABDE we get $DE \cdot AB + AE \cdot BD = AD \cdot BE$
which then becomes $DE \cdot AB = AD \cdot BE - AE \cdot BD$
3) $\Delta ADE \cong \Delta ADC$; $AE \cong CD$; Also $AC \cong DE$
4) $\angle BED \cong \angle BAD = \beta$;



5) $cos(\alpha) = \frac{AD}{AC} \rightarrow AD = AC\, cos(\alpha)$;
$sin(\beta) = \frac{BD}{DE} \rightarrow BD = DE\, sin(\beta) \rightarrow BD = AC\, sin(\beta)$;
$cos(\beta) = \frac{BE}{DE} \rightarrow BE = DE\, cos(\beta) \rightarrow BE = AC\, cos(\beta)$
$sin(\alpha) = \frac{CD}{AC} \rightarrow CD = AC\, sin(\alpha)$; $\rightarrow AE = AC\, sin(\alpha)$;
$cos(\alpha + \beta) = \frac{AB}{AC} \rightarrow AB = AC\, cos(\alpha + \beta)$
6) Substituting the values from #5 into the equation from #2 gives us:
$AC \cdot AC\, cos(\alpha + \beta) = [AC\, cos(\alpha)] \cdot [AC\, cos(\beta)] - [AC\, sin(\alpha)] \cdot [AC\, sin(\beta)]$
Dividing both sides by AC^2 we get:

$$\boxed{cos(\alpha + \beta) = cos(\alpha) \cdot cos(\beta) - sin(\alpha) \cdot sin(\beta)}$$

Table of Sines and Cosines

α	***sin*(α)**	***cos*(α)**	**α**	***sin*(α)**	***cos*(α)**	**α**	***sin*(α)**	***cos*(α)**	**α**	***sin*(α)**	***cos*(α)**
0.0°	0.00000	1.00000	22.5°	0.38268	0.92388	45.0°	0.70711	0.70711	67.5°	0.92388	0.38268
0.5°	0.00873	0.99996	23.0°	0.39073	0.92050	45.5°	0.71325	0.70091	68.0°	0.92718	0.37461
1.0°	0.01745	0.99985	23.5°	0.39875	0.91706	46.0°	0.71934	0.69466	68.5°	0.93042	0.36650
1.5°	0.02618	0.99966	24.0°	0.40674	0.91355	46.5°	0.72537	0.68835	69.0°	0.93358	0.35837
2.0°	0.03490	0.99939	24.5°	0.41469	0.90996	47.0°	0.73135	0.68200	69.5°	0.93667	0.35021
2.5°	0.04362	0.99905	25.0°	0.42262	0.90631	47.5°	0.73728	0.67559	70.0°	0.93969	0.34202
3.0°	0.05234	0.99863	25.5°	0.43051	0.90259	48.0°	0.74314	0.66913	70.5°	0.94264	0.33381
3.5°	0.06105	0.99813	26.0°	0.43837	0.89879	48.5°	0.74896	0.66262	71.0°	0.94552	0.32557
4.0°	0.06976	0.99756	26.5°	0.44620	0.89493	49.0°	0.75471	0.65606	71.5°	0.94832	0.31730
4.5°	0.07846	0.99692	27.0°	0.45399	0.89101	49.5°	0.76041	0.64945	72.0°	0.95106	0.30902
5.0°	0.08716	0.99619	27.5°	0.46175	0.88701	50.0°	0.76604	0.64279	72.5°	0.95372	0.30071
5.5°	0.09585	0.99540	28.0°	0.46947	0.88295	50.5°	0.77162	0.63608	73.0°	0.95630	0.29237
6.0°	0.10453	0.99452	28.5°	0.47716	0.87882	51.0°	0.77715	0.62932	73.5°	0.95882	0.28402
6.5°	0.11320	0.99357	29.0°	0.48481	0.87462	51.5°	0.78261	0.62251	74.0°	0.96126	0.27564
7.0°	0.12187	0.99255	29.5°	0.49242	0.87036	52.0°	0.78801	0.61566	74.5°	0.96363	0.26724
7.5°	0.13053	0.99144	30.0°	0.50000	0.86603	52.5°	0.79335	0.60876	75.0°	0.96593	0.25882
8.0°	0.13917	0.99027	30.5°	0.50754	0.86163	53.0°	0.79864	0.60182	75.5°	0.96815	0.25038
8.5°	0.14781	0.98902	31.0°	0.51504	0.85717	53.5°	0.80386	0.59482	76.0°	0.97030	0.24192
9.0°	0.15643	0.98769	31.5°	0.52250	0.85264	54.0°	0.80902	0.58779	76.5°	0.97237	0.23345
9.5°	0.16505	0.98629	32.0°	0.52992	0.84805	54.5°	0.81412	0.58070	77.0°	0.97437	0.22495
10.0°	0.17365	0.98481	32.5°	0.53730	0.84339	55.0°	0.81915	0.57358	77.5°	0.97630	0.21644
10.5°	0.18224	0.98325	33.0°	0.54464	0.83867	55.5°	0.82413	0.56641	78.0°	0.97815	0.20791
11.0°	0.19081	0.98163	33.5°	0.55194	0.83389	56.0°	0.82904	0.55919	78.5°	0.97992	0.19937
11.5°	0.19937	0.97992	34.0°	0.55919	0.82904	56.5°	0.83389	0.55194	79.0°	0.98163	0.19081
12.0°	0.20791	0.97815	34.5°	0.56641	0.82413	57.0°	0.83867	0.54464	79.5°	0.98325	0.18224
12.5°	0.21644	0.97630	35.0°	0.57358	0.81915	57.5°	0.84339	0.53730	80.0°	0.98481	0.17365
13.0°	0.22495	0.97437	35.5°	0.58070	0.81412	58.0°	0.84805	0.52992	80.5°	0.98629	0.16505
13.5°	0.23345	0.97237	36.0°	0.58779	0.80902	58.5°	0.85264	0.52250	81.0°	0.98769	0.15643
14.0°	0.24192	0.97030	36.5°	0.59482	0.80386	59.0°	0.85717	0.51504	81.5°	0.98902	0.14781
14.5°	0.25038	0.96815	37.0°	0.60182	0.79864	59.5°	0.86163	0.50754	82.0°	0.99027	0.13917
15.0°	0.25882	0.96593	37.5°	0.60876	0.79335	60.0°	0.86603	0.50000	82.5°	0.99144	0.13053
15.5°	0.26724	0.96363	38.0°	0.61566	0.78801	60.5°	0.87036	0.49242	83.0°	0.99255	0.12187
16.0°	0.27564	0.96126	38.5°	0.62251	0.78261	61.0°	0.87462	0.48481	83.5°	0.99357	0.11320
16.5°	0.28402	0.95882	39.0°	0.62932	0.77715	61.5°	0.87882	0.47716	84.0°	0.99452	0.10453
17.0°	0.29237	0.95630	39.5°	0.63608	0.77162	62.0°	0.88295	0.46947	84.5°	0.99540	0.09585
17.5°	0.30071	0.95372	40.0°	0.64279	0.76604	62.5°	0.88701	0.46175	85.0°	0.99619	0.08716
18.0°	0.30902	0.95106	40.5°	0.64945	0.76041	63.0°	0.89101	0.45399	85.5°	0.99692	0.07846
18.5°	0.31730	0.94832	41.0°	0.65606	0.75471	63.5°	0.89493	0.44620	86.0°	0.99756	0.06976
19.0°	0.32557	0.94552	41.5°	0.66262	0.74896	64.0°	0.89879	0.43837	86.5°	0.99813	0.06105
19.5°	0.33381	0.94264	42.0°	0.66913	0.74314	64.5°	0.90259	0.43051	87.0°	0.99863	0.05234
20.0°	0.34202	0.93969	42.5°	0.67559	0.73728	65.0°	0.90631	0.42262	87.5°	0.99905	0.04362
20.5°	0.35021	0.93667	43.0°	0.68200	0.73135	65.5°	0.90996	0.41469	88.0°	0.99939	0.03490
21.0°	0.35837	0.93358	43.5°	0.68835	0.72537	66.0°	0.91355	0.40674	88.5°	0.99966	0.02618
21.5°	0.36650	0.93042	44.0°	0.69466	0.71934	66.5°	0.91706	0.39875	89.0°	0.99985	0.01745
22.0°	0.37461	0.92718	44.5°	0.70091	0.71325	67.0°	0.92050	0.39073	89.5°	0.99996	0.00873
									90.0°	1.00000	0.00000