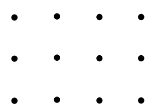
Puzzles - for 10th grade workshop

1. Three Men

There are three men – Don, Ron and Lon – two of whom are married, two have brown eyes, and two are bald. The one with hair has blue eyes. Don's wife is Ron's sister. The bachelor and Lon have the same color eyes. Which man has hair?

2. Connect-the-Dots Rectangle

Without lifting your pencil off the page, and ending up back at the place where you started, draw five lines that pass through all 12 of the points in the 4-by-3 grid shown here.



3. Crossing a Desert

You are on the edge of a desert that is 800 miles wide. There is an unlimited supply of fuel at the start. Your truck can carry enough fuel (in its gas tank and in storage tanks) to travel exactly 500 miles. At any point along the route, you may remove as much fuel as you like from your truck, put it in storage tanks, and leave it in the desert.

a) In order to cross the desert, what is the minimum number of times needed to return to the start to pick up more fuel?

b) What is the largest desert that can be crossed?

4. Missing-Digit Multiplication

Fill in the missing digits (indicated by "?") of these problems.

```
     ???
    x 74
    ----
    2152
 + ????0
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   ?????
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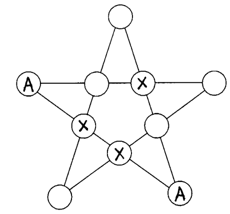
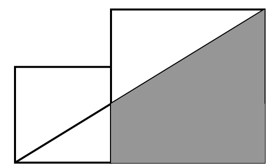
5. A's and X's

The figure here must be filled in such that each row of four circles contains two A's and two X's. Which circle <u>must</u> be filled with an A?



6. A Shaded Region

Given that the two squares, shown here, have sides of length 8 and 5, find the area and perimeter of the shaded region.



7. Siblings

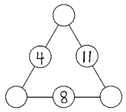
John and Emily are siblings. John has five times as many sisters as brothers, and Emily has three times as many sisters as brothers. How many children are in the family?

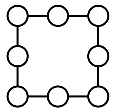
8. Clock Hands

When are the minute and hour hand of a clock *exactly* together between four and five o'clock?

9. A Square and a Triangle

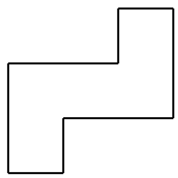
With each figure below, fill in the circles so that each row of three circles have the same sum. (Each circle must be filled in with a different number.)

a) 

b) 

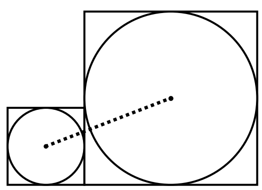
10. Making a Square

The figure shown on the right has only right angles, and each edge has a length of either 1 inch or 2 inches. How can you make two straight cuts such that the three resulting pieces can be arranged to form one square?



11. Connected Circles

The squares, shown here, have sides of length 14 and 34. What is the length of the line that joins the centers of the inscribed circles?



12. Missing-Digit Multiplication

Fill in the missing digits (indicated by "?") of these problems.

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     ?9
   x ??
    ?77
 + 4??0
   ?30?
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13. The Three Daughters

A clever man says to a clever girl, "I have three daughters. The product of their ages is 72, and the sum of their ages is the same as your age."

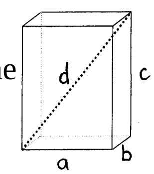
The girl sits down and thinks about it. After a while she says, "I still need more information."

"OK," says the man. "My oldest daughter is shorter than you." Then the girl quickly and correctly gives the ages of the man's three daughters.

What is the age of each daughter?

14. Pythagorean Quadruples

Pythagorean triples are the three sides of a right triangle, where all three sides (a, b, c) work out as whole numbers. The well-known formula that relates these three lengths is $c^2 = a^2 + b^2$. We can instead relate this to a rectangle, where a and b are the lengths of the sides of the rectangle, and c is the length of its diagonal. Now, taking this idea into three dimensions, we have a right rectangular prism (i.e., a box), where a, b and c are the length, width and height of the prism, and d is the body diagonal of the prism. We have a Pythagorean quadruple when all four lengths turn out to be whole numbers.



a) Find the length of the diagonal of a prism that has dimensions (a, b, c) equal to 6, 9, and 12. (This should yield a Pythagorean quadruple.)
b) Give a formula that relates a, b, c, and d.
c) Find as many Pythagorean quadruples as you can.

15. Five Hats

Five boys (A, B, C, D, E) each with a hat on that is either yellow or red, stand in a circle looking at one another. Each boy cannot see the color of his own hat, but can see the color of all the other hats.

Boy A says: "I see three yellow hats and one red hat."
Boy B says: "I see four red hats."
Boy C says: "I see three red hats and one yellow hat."
Boy E says: "I see four yellow hats."

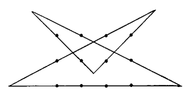
Determine the color of each boy's hat given that any boy with a red hat always tells the truth, and any boy with a yellow hat only tells lies.

Solutions for 10^{th} Grade Puzzles

1. Three Men

The statement "the bachelor and Lon have the same colored eyes" tells us that Lon must be married and has brown eyes. Don is also married, so Ron must be the bachelor, and therefore must have brown eyes. So Don must have blue eyes, and <u>Don has hair</u>.

2. Connect the Dots Rectangle



3. Crossing a Desert

a) There are many strategies that will get you 800 miles on 4 fill-ups. Here is one.

Trip 1: Drive 100 miles into the desert and drop 300 miles of fuel. Return to start.

Trip 2: Drive 100 miles into the desert. Pick up 100 miles of fuel. Drive to mile 200. Drop off 200 miles of fuel. Return to start.

Trip 3: Drive 100 miles into the desert. Pick up 100 miles of fuel. Drive to mile 200. Pick up 100 miles of fuel. Drive to mile 300. Drop off 100 miles of fuel. Return to start.

Trip 4: Drive straight through picking up 100 miles of fuel at mile 100, 200, and 300.

b) With an infinite supply of fuel, infinite supply of gas cans, and an infinite amount of time, any finite desert could be crossed.

4. Missing-Digit Multiplication

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a)     538
      x 74
      ----
      2152
  + 37660
  -------
     39812
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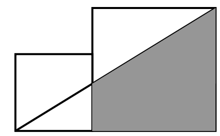
5. A's and X's

It is helpful to reframe the question, and instead ask ourselves, "What circles <u>can't</u> be an X?" We can then see that only the <u>bottom-left circle</u> can't be assigned an X, for that would lead to needing one row with three X's or three A's.

6. A Shaded Region



Let x be the vertical height of the left side of the shaded region, and let y be the length of the slanted top of the shaded region. Using similar triangles, we can set up the ratio 5:x = 13:8, which yields an x value equal to $^{40}/_{13} \approx 3.08$. Using the Pythagorean Theorem allows us to calculate that y is $\approx$ 9.39. We can then easily calculate that <u>the area of the shaded region is ≈ 44.3, and the perimeter is ≈ 28.5</u>.

7. Siblings

There are <u>13 children in the family</u> (10 girls and 3 boys).

8. Clock Hands

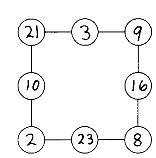
One way to approach this problem is by imagining that at 4:00 a race begins where the minute hand sets off to catch up with the hour hand. Since there are 60 tick marks on the clock, we can set our standard of measurement to one "tick mark", and say that the minute hand is moving at a rate of 60 ticks/hr and the hour hand is moving at a rate of 5 ticks/hr. The minute hand is therefore catching up at a rate of 55 ticks/hr. Given that the minute hand started out 20 tick marks behind the hour hand, the amount of time for the minute hand to catch up is 20 ticks ÷ 55 ticks/hr, which is $^{4}/_{11}$ of an hour, or $21\,^{9}/_{11}$ minutes. Therefore, the hands are exactly together at <u>$4{:}21\,^{9}/_{11}$</u>, which is <u>21 minutes 49 $^{1}/_{11}$ seconds (or ≈49.1 seconds) after 4 o'clock</u>.

9. A Square and a Triangle

There are infinitely many possible solutions for both of these.

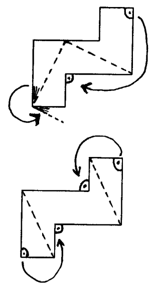
a) Choose any number for the top circle. The bottom-left circle must then be 3 greater than the top circle, and the bottom-right circle must be 4 less than the top circle.

b) In this case, the key is to make it so that the two circles falling on the vertical midline (shown here as 23 and 3) have the same sum as the two circles falling on the horizontal midline (shown here as 10 and 16).



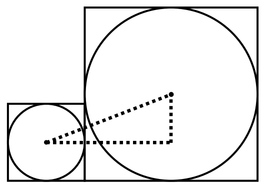
Solutions

10. Making a Square



It is helpful here to realize that the area of the square must be 5, so the length of the side of the square must be $\sqrt{5}$. So then the question becomes, "How can we make a line of length $\sqrt{5}$?" Well, this is the length of the hypotenuse of a right triangle that has legs of length 2 and 1. Two possible solutions are shown.

11. Connected Circles



The key is to construct a right triangle (as shown here) where the horizontal leg has a length equal to the sum of the circles' radii, and the vertical leg has a length equal to the difference of the circles' radii. The legs are therefore 24 and 10. Either using the Pythagorean Theorem, or recognizing that these numbers are in a Pythagorean triple ratio (5-12-13), tells us that the line connecting the circles' centers has a length of <u>26</u>.

12. Missing-Digit Multiplication

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    59
  x 73
   177
+ 4130
  4307
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13. The Three Daughters

Because knowing her own age wasn't enough information, we know that there must be two sets of (three) numbers that both multiply to 72 and add to whatever the girl's age is. If we then list all of the sets of three numbers whose product is 72 (e.g., 1, 1, 72 and 1, 2, 36, etc.) then we will find that only the following two sets have the same sum: 2, 6, 6 and 3, 3, 8. Most notably, each set shows that two of the three daughters are, in fact, twins. Therefore when the man speaks of his "oldest daughter", the girl can then conclude that 2, 6, 6 aren't the ages because, if that were so, there wouldn't be an "oldest" daughter. So the answer must be that <u>the daughters are 3, 3 and 8</u>.

14. Pythagorean Quadruples

a) 17

b) $d^2 = a^2 + b^2 + c^2$

c) One possible method is to rewrite the formula as $a^2 + b^2 = d^2 - c^2$ and then as $a^2 + b^2 = (d - c)(d + c)$. Now, choose any Pythagorean triple (not necessarily reduced), and assign a and b to the legs of that triple, and let x be the hypotenuse of that triple. This means that $a^2 + b^2 = x^2$ and that $x^2 = (d - c)(d + c)$. Now find two numbers (either both odd if x^2 is odd, or both even if x^2 is even) whose product is equal to x^2. c will then be half the distance between the two numbers, and d will the average of the two numbers.

For example, choosing the Pythagorean triple 12, 16, 20, gives us a=12, b=16, x=20. x^2 is then 400. Choosing 8 and 50 as the two numbers that multiply to 400, gives us c=21, and d=29 for a Pythagorean quadruple (12,16,21,29). Choosing differently from 8 and 50 would have yielded the quadruples (12,16,99,101), (12,16,15,25), and (12,16,48,52), which reduces to (3,4,12,13). Some other Pythagorean quadruples: are (1,2,2,3); (2,3,6,7); (4,4,7,9); (9,12,20,25); (9,12,112,113).

15. Five Hats

The key to this problem is to seek out the contradictions and let them make the determination for you. This can be solved by the process of elimination, but you need a place to start.

Boy B could not see 4 red hats, because that would make all the other boys truth-tellers, but their statements contradict this. *So Boy B is lying and has a yellow hat.*

Boy C cannot be telling the truth. For if he were, then Boy E would also have to be telling the truth (since we already have determined that Boy B has a yellow hat), but then Boy E's statement wouldn't work. *So Boy C is lying and has a yellow hat.*

If Boy E were telling the truth, then Boy A would be seeing three yellow hats and a red hat, but that would mean Boy A would also be telling the truth, which is a contradiction. *So Boy E is lying and has a yellow hat.*

Now we need only to determine the color of the hats for Boy A and Boy D. They can't both be yellow, because that would make Boy E a truth-teller. The only thing that works is if Boy A and Boy D both have red hats.

Thus: <u>**A**: red; **B**: yellow; **C**: yellow; **D**: red; **E**: yellow</u>.

Lastly, we can confirm that this solution works by checking that all of the statements are consistent with the boys' hat colors.