Proof of Heron's Formula

for the Area of a Triangle

Area ΔABC = $\sqrt{s(s-a)(s-b)(s-c)}$, where the semi-perimeter, s = ½(a + b + c).

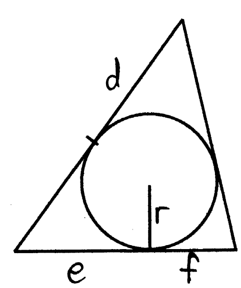
Background theorems:

- *Inscribed Quadrilateral Theorem* (Elements Th. III-22). The opposite angles of a quadrilateral inscribed inside a circle are supplementary.
- *Proportionality Laws*
  - *Elements Th. V-16*. If a:b = c:d then a:c = b:d.
    Example: If 6:9 = 8:12 then 6:8 = 9:12
  - *Elements Th. V-18*. If a:b = c:d then (a+b):b = (c+d):d [also a:(a+b) = c:(c+d)].
    Example: If 6:9 = 8:12 then 15:9 = 20:12
- *The Altitude of the Hypotenuse Theorem* (Elements Th. VI-8).
  If the altitude to the hypotenuse of a right triangle is drawn, then the two triangles formed are similar to the original triangle and to each other, and the square of the length of the altitude is equal to the product of the lengths of the two segments of the hypotenuse.
- **Theorem A:** In a quadrilateral, if the two diagonals and two opposite sides form two angles that are equal, then the quadrilateral is *cyclic* (all four vertices lie on a circle).

  Proof:
  1. Given quadrilateral ABCD where ∠ABD ≅ ∠ACD, draw a circle through points A, B, D (the circumcenter of ΔABD).
  2. *Assume point C is NOT on the circle*. Point E is the intersection of line AC and the circle.
  3. ∠AED≅∠ABD (Elements Th. III-21: "*Inscribed angles that are subtended by the same arc are equal to one another.*")
  4. ∠ACD≅∠AED (C.N. 1; Transitive Property)
  5. If ∠ACD ≅ ∠AED, then CD is parallel to ED (Corresponding Angle Th.)
  6. If CD is parallel to ED then they don't intersect. (Def of parallel)
  7. But CD *does* intersect ED at point D, so we have a contradiction.
  8. ∴Our assumption is false; therefore point C must also lie on the circle. The quadrilateral is cyclic.

- **Theorem B:** Area of a triangle = r · s; where r is the radius of the inscribed circle, and s is the semi-perimeter of the triangle.

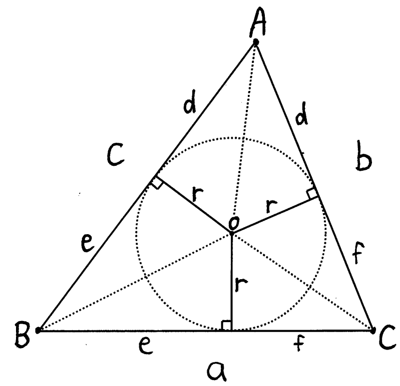
  Proof: Draw angle bisectors through all three angles of the triangle. Using the point where the three bisectors intersect as the center, draw the circle that is inscribed in the triangle. From the center (I) of the circle, draw lines to the vertices of the triangle, and to the points of tangency. This creates six small triangles. Since the vertices of the original triangle are bisected, and the newly drawn lines to the tangent points form right angles, then the small triangles that share hypotenuses are congruent. Six of the sides of these triangles are segments of the sides of the original triangle. These segments are equal in three pairs, and these three lengths we shall call d, e, and f. The area of the original triangle is therefore:



  Area = 2(½d·r) + 2(½e·r) + 2(½f·r)
  Area = r (d + e + f)
  Area = r · s, where s is the semi-perimeter.

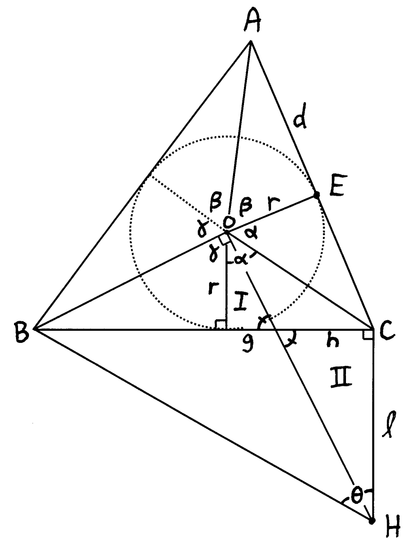
Heron's Formula[1] (continued)

<u>The Proof of Heron's Formula</u>:



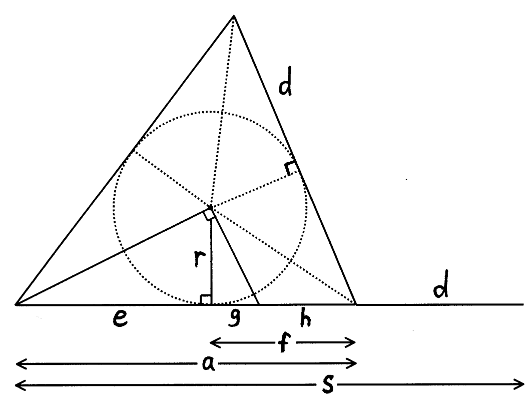
Given any triangle with points A, B, C, and sides a, b, c…

1. **The first drawing:** Draw the inscribed circle of ΔABC with O as center. Draw perpendicular lines from O to the three sides of the triangle, and draw lines from O to A, B, C. Label sides as shown.
2. **The second drawing:** Draw a line perpendicular to BO from O, and another line perpendicular to BC from C, and label their intersection as H. Draw OH and BH. Label sides and angles as shown.
3. **Unexpected Similar Triangles:**
   a. Because the original triangle is divided into three pairs of congruent triangles, and because the sum of angles surrounding the center of the circle is 360°, we can say that $\alpha + \beta + \gamma = 180°$.
   b. Since the sides of quadrilateral BOCH and its diagonals form equal angles ($\angle BOH$ and $\angle BCH$), we know that BOCH is cyclic (fits inside a circle). (*Theorem A* from "Background Theorems")
   c. The *Inscribed Quadrilateral Theorem* tells us that opposite angles add to 180°. Therefore, $\angle BOC + \angle BHC = 180° \rightarrow (\alpha + \gamma) + \theta = 180°$.
   d. Steps #3a and #3c allow us to say that $\boldsymbol{\theta = \beta}$
   e. Congruent angles give us $\Delta AOE \sim \Delta BHC$ $\ell : a = r : d \rightarrow \ell : r = a : d$
   f. Also triangles I and II are similar, which gives us $\ell : h = r : g \rightarrow \ell : r = h : g \rightarrow$ <u>**$a : d = h : g$**</u>



4. **The third drawing:** Extend side BC past C a distance equal to length d.. Label sides as shown. Note that S is the semi-perimeter: $S = e + f + d$

5. **Bringing it all Together:**
   a. From step #3f, note that a, d, h, g are all on the same line. Using *Elements Th. V-18* gives us: $a : d = h : g \rightarrow s : d = f : g \rightarrow g \cdot s = d \cdot f$
   b. Altitude of the Hypotenuse Th tells us: $r^2 = g \cdot e$
   c. $g \cdot s = d \cdot f \rightarrow$ mult by e $\rightarrow g \cdot e \cdot s = d \cdot e \cdot f$
   $\rightarrow r^2 \cdot s = d \cdot e \cdot f \rightarrow$ mult by s $\rightarrow r^2 \cdot s^2 = s \cdot d \cdot e \cdot f$
   $\rightarrow r \cdot s =$ Area of $\Delta ABC = \sqrt{s(d)(e)(f)}$
   (*Theorem B* from "Background Theorems")



   d. We know that: $s = e + f + d$ and that $a = e + f$
   Therefore $s = a + d \rightarrow$ <u>$d = s - a$</u>
   Likewise <u>$e = s - b$</u> and <u>$f = s - c$</u>

   e. Substituting into the above equation gives us **Area of $\Delta ABC = \sqrt{s(s-a)(s-b)(s-c)}$**

Q.E.D.

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1. The key steps are from *an Introduction to the History of Mathematics* by Howard Eves, page 172. Further improvements were made and inspired by Tomas Campomanes.