The Foundation of Euclid's Elements

The 34 Definitions (some of the names of the terms have been changed from Heath's original translation.)

1. A **point** is that which has no part.
2. A **curve** is breadthless length.
3. The **extremities of a curve** are points.
4. A **line** is a curve which lies evenly with the points on themselves.
5. A **surface** is that which has length and breadth only.
6. The **extremities of a surface** are curves.
7. A **plane** is a surface which lies evenly with the lines on themselves.
8. A plane **angle** is the inclination to one another of two curves in a plane which meet one another and do not lie in a straight line.
9. When the sides of an angle are straight lines, then the angle is called **rectilineal**.
10. When a line stood up on another line makes the adjacent angles equal to one another, each of the angles is **right**, and the line standing on the other is called a **perpendicular** to that on which it stands.
11. An **obtuse angle** is an angle greater than a right angle.
12. An **acute angle** is an angle less than a right angle.
13. A **boundary** is that which is an extremity of anything.
14. A **figure** is that which is contained by any boundary or boundaries.
15. A **circle** is a plane figure contained by one curve such that all the lines falling upon it from one point among those lying within the figure are equal to one another.
16. And the point is called the **center** of the circle.
17. A **diameter** of the circle is any line drawn through the center and terminated in both directions by the circumference of the circle, and such a line also bisects the circle.
18. A **semi-circle** is the figure contained by the diameter and the circumference cut-off by it. The center of the semi-circle is the same as that of the circle.
19. **Rectilineal figures** are those which are contained by lines. **Triangles** are those contained by three lines, **quadrilaterals** are those contained by four lines, and **multilaterals** are those contained by more than four lines.
20. Of triangles, an **equilateral triangle** has three equal sides; an **isosceles triangle** has two of its sides alone equal; a **scalene** triangle has its three sides unequal, and…
21. A **right triangle** has a right angle; an **obtuse triangle** has an obtuse angle; and an **acute triangle** has its three angles acute.
22. Of quadrilaterals, a **square** is both equilateral and right-angled; an **oblong** is right-angled but not equilateral [i.e., a rectangle which is not a square]; a **rhombus** is equilateral but not right-angled; and a **rhomboid** has its opposite sides and angles equal to one another, but is neither equilateral nor right-angled [i.e., a parallelogram which is neither a rhombus nor a rectangle]. Let all other quadrilaterals be called **trapezia.**
23. **Parallel** lines are lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

The 5 Common Notions

1. <u>Transitive Property</u>. Things which are equal to the same thing are also equal to each other.
2. <u>Addition Property</u>. If equals are added to equals, then the sums are equal.
3. <u>Subtraction Property</u>. If equals are subtracted from equals, then the differences are equal.
4. Things (i.e. figures or solids) that coincide with one another are congruent.
5. The whole is greater than the part.

The 5 Postulates

1. A line can be drawn between any two given points.
2. Any line can be extended.
3. A circle can be drawn with any center and any distance [as its radius].
4. All right angles are equal.
5. <u>The Parallel Postulate</u>. If a line falling on two lines makes the interior angles on the same side less than two right angles, the two lines, if produced indefinitely, meet on that side on which the angles are less than two right angles.

A Summary of

The 13 Books of Euclid's Elements

Book I Basic constructions and theorems involving angles, triangles, parallelograms, etc.

Th. 1: To construct an equilateral triangle.
Th. 4: SAS triangle congruency theorem.
Th. 5: Isosceles Triangle Theorem.
Th. 8: SSS triangle congruency theorem.
Th. 13: Supplementary Angle Theorem.
Th. 15: Vertical Angle Theorem.
Th. 26: AAS, AAS triangle congruency.
Th. 29: (a) Corresponding angles are equal.
(b) Alternate interior angles are equal.
(c) Same-side interior angles are equal.
Th. 32: The angles in a triangle add to 180°.
Th. 35-38, 41: Shear and Stretch theorems.
Th. 46: To construct a square given one side.
Th. 47: Pythagorean Theorem.

Book II "Geometrical Algebra" and theorems regarding equal area.

Th. 4: "*If a straight line be cut at random, the square on the whole is equal to the squares on the segments and twice the rectangle contained by the segments.*" This is the geometric equivalent of the identity $(a+b)^2 = a^2 + 2ab + b^2$.
Th. 5: Geometric equivalent of the identity $(a+b)(a-b) = a^2 - b^2$, and used as solution to $ax - x^2 = b^2$.
Th. 11: "*How to cut a given straight line so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining side.*" This is the equivalent of solving $ax + x^2 = a^2$.
Th. 12, 13: Variation of the Pythagorean Theorem for acute and obtuse triangles. (Law of Cosines)
Th. 14: Construction of a square with an area equal to a given polygon.

Book III Circle geometry

Th. 1: Finding the center of a given circle.
Th. 17: Constructing a tangent line through a given point on a circle.
Th. 20: Inscribed Angle Theorem.
Th. 21: "*Inscribed angles that are subtended by the same arc are equal to one another.*"
Th. 22: Inscribed Quadrilateral Theorem.
Th. 31: Theorem of Thales.
Th. 32: Chord-Tangent Theorem.
Th. 35: Chord-Segment Theorem.
Th. 36: Secant-Tangent Theorem.
Th. 37: Secant-Tangent Theorem converse

Book IV Constructions of polygons

Th. 1: To draw a chord inside a given circle equal to a given line.
Th. 2,3: To inscribe or circumscribe in or about a given circle a triangle similar to a given triangle.
Th. 4,5: To inscribe or circumscribe a circle in or about a given triangle.
Th. 6-9: To inscribe or circumscribe a square in or about a given circle, and vice versa.
Th. 10: To inscribe a *golden triangle* (the triangle found in a regular pentagon) in a given circle.
Th. 11-14: To inscribe or circumscribe a regular pentagon in or about a given circle, and vice versa.
Th. 15: To inscribe a regular hexagon in a given circle.
Th. 16: To inscribe a regular 15-gon in a given circle.

Book V The "algebra" of proportions independent of geometry. (Eudoxus' theory of proportion)

Th. 1: States $ma+mb+mc+... = m(a+b+c...)$.
Th. 4: States that if $a:b = c:d$ then $ma:nb = mc:nd$.
Th. 5: States $ma-mb = m(a-b)$.
Th. 9: States that if $a:c = b:c$ then $a=b$.
Th. 10: States that if $a:c > b:c$ then $a>b$.
Th. 11: States that if $a:b = c:d$ and $c:d = e:f$ then $a:b = e:f$.
Th. 12: States that $a:a' = b:b' = c:c'.... = (a+b):(a'+b') = (a+b+c...):(a'+b'+c'...)$.
Th. 15: States $a:b = ma:mb$.
Th. 16: States that if $a:b = c:d$ then $a:c = b:d$.
Th. 17: States that if $a:b = c:d$ then $(a-b):b = (c-d):d$.
Th. 18: States that if $a:b = c:d$ then $(a+b):b = (c+d):d$ [also $a:(a+b) = c:(c+d)$].
Th. 19: States that if $a:b = c:d$ then $a:b = (a-c):(b-d)$.
Th. 25: States that if $a:b = c:d$ and a is the greatest and d is the least, then $a+d > b+c$

Book VI Proportions relating to geometry.

Th. 2: Triangle proportionality theorem.
Th. 3: Triangle angle-bisector theorem.
Th. 4: Similar triangles have their sides in equal proportions.
Th. 5: SSS similarity theorem.
Th. 6: SAS similarity theorem.
Th. 7: HL similarity theorem.
Th. 8: Altitude of the hypotenuse theorem.
Th. 11: To construct the third term in a geometric progression. (Given x and y, find z such that x:y = y:z).
Th. 12: Given a, b, c to construct x such that a:b = c:x.
Th. 13: To construct the geometric mean. (Given x, z find y such that x:y = y:z).
Th. 16: "*If four lines are proportional, then the rectangle contained by the extremes is equal to the rectangle contained by the means [and vice versa].*" Algebraically, this is: if a:b = c:d then a•d = b•c.
Th. 25: To construct a polygon similar to one polygon and equal in area to another polygon.
Th. 28, 29: Geometric solution to a general quadratic equation.
Th. 30: To cut a given line into the *Golden Ratio* (golden section).
Th. 31: Generalization of the Pythagorean Theorem. If a right triangle has similar figures drawn in the same orientation off each of its three sides, then the area of the largest figure equals the sum of the areas of the other two figures.
Th. 33: With two equal circles, angles have the same ratio as the arcs that subtend them, whether they stand at the centers or at the circumferences.

Book VII, VIII, IX Elementary number theory.

Th. VII-2: To find the greatest common factor of two numbers.
Th. VII-16: Commutative property of multiplication a•b = b•a
Th. VII-32: Any number is either prime or has some prime number as its factor.
Th. VII-34: To find the least common multiple of two numbers.
Th. VIII-14: If a^2 is a factor of b^2, then a is a factor of b, and vice versa.
Th. VIII-16: If a^2 is *not* a factor of b^2, then a is *not* a factor of b, and vice versa.
Th. VIII-24: If $a:b = c^2:d^2$ and c is a square, then d is a square.
Th. VIII-25: If $a:b = c^3:d^3$ and c is a cube, then d is a cube.
Th. IX-14: *The Fundamental Theorem of Arithmetic*, which says that any integer greater than one can be expressed as the product of primes in only one way. (e.g. $24 = 2^3 \cdot 3$)
Th. IX-20: States that the number of primes is infinite.
Th. IX-35: The equivalent of a formula for the sum of the numbers in a geometric progression.
Th. IX-36: *Euclid's Perfect Number Theorem.* If $2^n - 1$ is prime, then $2^{n-1}(2^n - 1)$ is perfect.

Book X On Irrational Numbers (115 theorems!)

Th. X-3: Given two commensurable magnitudes, to find their common measure (common multiple).
Th. X-28: A method for determining Pythagorean triples.
Th. X-42: The equivalent of saying if $a+\sqrt{b} = x+\sqrt{y}$ then a=x and b=y.
Th. X-112: The equivalent of rationalizing the denominator of a fraction – e.g. $\frac{3}{\sqrt{7}+\sqrt{5}}$.

Book XI. XII, XIII Solid geometry.

Th. XII-2: *(The Basis for the Method of Exhaustion)* The areas of two circles have the same ratio as the areas of the squares of their diameters.
Th. XII-7,10: The formula for the volume of a *pyramid* and a *cone*.
Th. XIII-10: A triangle whose sides are respectively the sides of an equilateral pentagon, hexagon, and decagon, all inscribed in the same circle, is a right triangle.
Th. XIII-13,14,15,16,17: The ratio of the edges of a tetrahedron to an octahedron to a cube to an icosahedron to a dodecahedron to the diameter of the sphere that circumscribes them all is:

$$T : O : C : I : D : S = \sqrt{2/3} : \sqrt{1/2} : \sqrt{1/3} : \sqrt{\frac{5-\sqrt{5}}{10}} : \sqrt{\frac{\sqrt{5}-1}{2\sqrt{3}}} : 1$$

Th. XIII-18: There are only five possible regular [Platonic] solids.