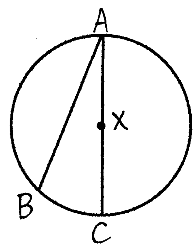
Standard Proofs for
The Inscribed Angle Theorem
and
The Intersecting Chord Theorem

Standard Proof of <u>*Inscribed Angle Theorem*</u>:

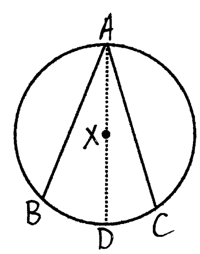
- **All three of these cases need to be proven:**

Case #1 A side of the angle is a diameter of the circle.



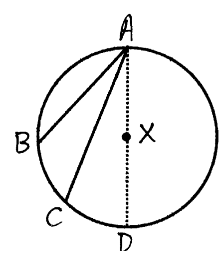
1. Draw segment BX.
2. $\angle BXC = \angle XAB + \angle XBA$
3. $\angle XAB \cong \angle XBA$
4. $\angle XAB = \frac{1}{2}\angle BXC$
5. $\therefore \angle BAC = \frac{1}{2}\angle BXC = \frac{1}{2}\text{arc}BC$

Case #2 The center lies in the interior of the angle.



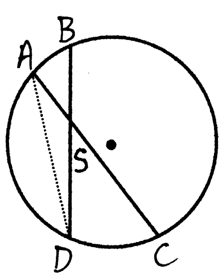
1. Draw diameter from A through X to point D.
2. $\angle BAC = \angle BAD + \angle DAC$
3. $\angle BAD = \frac{1}{2}\angle BXD$; $\angle DAC = \frac{1}{2}\angle DXC$
4. $\angle BAC = \frac{1}{2}\angle BXD + \frac{1}{2}\angle DXC$
5. $\angle BAC = \frac{1}{2}(\angle BXD + \angle DXC)$
6. $\angle BXC = \angle BXD + \angle DXC$
7. $\therefore \angle BAC = \frac{1}{2} \angle BXC = \frac{1}{2}\text{arc}BC$

Case #3 The center lies outside of the angle.



1. Draw diameter from A through X to point D.
2. $\angle BAD = \angle BAC + \angle DAC$
3. $\angle BAD = \frac{1}{2}\angle BXD$; $\angle DAC = \frac{1}{2}\angle DXC$
4. $\frac{1}{2}\angle BXD = \angle BAC + \frac{1}{2}\angle DXC$
5. $\angle BAC = \frac{1}{2}(\angle BXD - \angle DXC)$
6. $\angle BXC = \angle BXD - \angle DXC$
7. $\therefore \angle BAC = \frac{1}{2} \angle BXC = \frac{1}{2}\text{arc}BC$

Standard Proof of <u>*Intersecting Chord Theorem*</u>

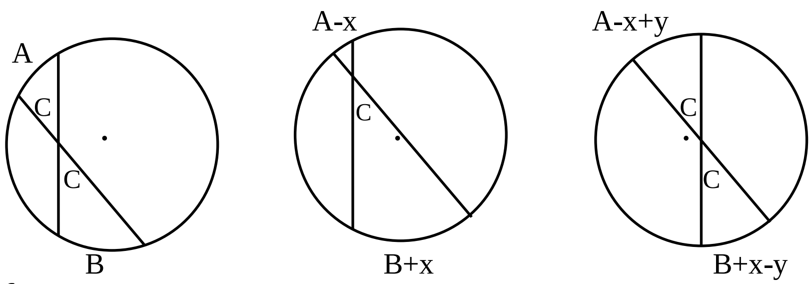


1. Draw chord AD
2. $\angle A = \frac{1}{2}\text{arc}DC$
3. $\angle D = \frac{1}{2}\text{arc}AB$
4. $\angle S = \angle A + \angle D$ (Δ exterior angle Th.)
5. $\angle S = \frac{1}{2}\text{arc}DC + \frac{1}{2}\text{arc}AB$
6. $\angle S = \frac{1}{2}(\text{arc}DC + \text{arc}AB)$

Movement Proofs for
The Inscribed Angle Theorem
and
The Intersecting Chord Theorem

Intersecting Chord Theorem.

"The measure of an angle formed by two chords that intersect inside a circle is equal to the arithmetic mean of the measures of the two intercepted arcs."



Proof:

1. Given two intersecting chords, which form angle C and the two subtended arcs A and B.
2. Slide one chord over to the center of the circle such that the new position of the chord is parallel to the original. One arc shrinks by the same amount (x°) that the other arc grows (parallel chord theorem). The two arcs may now be labeled $A-x$ and $B+x$. Note also that angle C has not changed (corresponding angles).
3. Slide the other chord also over to the center in the same manner, thereby shrinking and enlarging the two arcs by the same amount (y°). The two arcs are now $A-x+y$ and $B+x-y$. (Note that if angle C subtended the center of the circle then the two arcs would now have be: $A-x-y$ and $B+x+y$)
4. Since angle C is now a central angle, both of the resulting arcs are now equal to angle C. Therefore we can write two equations and then add them together to get:

$$\begin{array}{rl} C &= A-x+y \\ C &= B+x-y \\ \hline 2C &= A + B \end{array}$$

Added together gives: $2C = A + B$ or $\longrightarrow$ $C = \frac{1}{2}(A + B)$

Inscribed Angle Theorem.

"The measure of an inscribed angle is half the measure of its intercepted arc."

Proof: Simply imagine two intersecting chords where the point of intersection moves to edge of the circle. One of the two arcs has now become zero. Using the intersecting chord theorem we now get: $C = \frac{1}{2}(A + B) \longrightarrow C = \frac{1}{2}(A + 0) \longrightarrow C = \frac{1}{2}A$

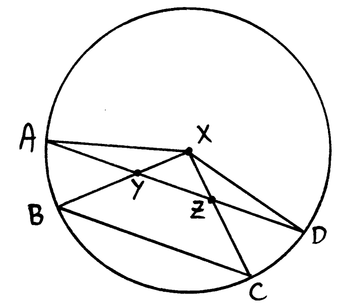
- *Corollary:* Inscribed angles that intercept equal arcs, are congruent.
- *Corollary:* Theorem of Thales. An angle inscribed in a semicircle is a right angle. (Review from Greek geometry main lesson).

Proof of the **<u>Parallel Chord Theorem</u>**

"Two arcs that lie between two parallel chords are congruent."

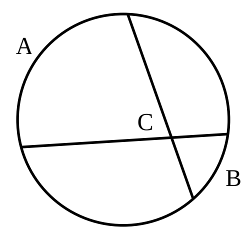
Proof: (Case #1: Both chords to the same side of the center.)

1. Given 2 parallel chords, draw radii to the 4 points of intersection.
2. $\angle XBC \cong \angle XCB$ (Isosceles Δs)
   $\angle XYZ \cong \angle XBC$ and $\angle XZY \cong \angle XCB$ (Corresp. $\angle$s)
   $\angle XYZ \cong \angle XZY$ (Transitive)
3. $\angle XAY \cong \angle XDZ$ (Isosceles Δs)
4. $\angle XYZ = \angle XAY + \angle AXB$ (Exterior $\angle$s)
   $\angle XZY = \angle XDZ + \angle DXC$ (Exterior $\angle$s)
5. $\angle AXB = \angle XYZ - \angle XAY$
   $\angle DXC = \angle XZY - \angle XDZ = \angle XYZ - \angle XAY$ (subst.)
6. $\angle AXB \cong \angle DXC$ and $\therefore$ **arcAB $\cong$ arcCD**

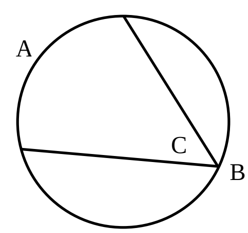


Proof of the **<u>Outside Angle Theorem</u>**

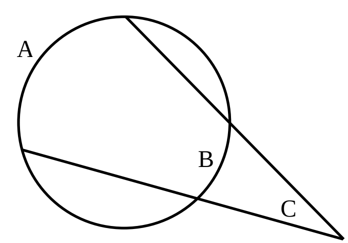
"The measure of an angle formed by two secants, or two tangents, or a secant and a tangent, that intersect each other outside the circle is equal to half the difference of the measures of the intercepted arcs."



Movement Proof: We will do the same as with our movement proof for the inscribed angle theorem. Start with the case of the angle formed by two intersecting chords. The formula for calculating this angle is C = ½(A + B). Now imagine that the vertex of this angle (see top drawing) moves away from the center of the circle until it lies on the edge of the circle. Arc B is now zero and the formula is C = ½A. (Now the tricky part!) Allow the vertex of the angle to continue to move away from the center of the circle, thereby creating the intersection of two secants that intersect outside the circle. Arc B had a positive value when the vertex was inside the circle and then a value of zero when it was on the circle. Now that the vertex is outside the circle, it can be imagined that *arc B has a negative value*. The formula for an outside angle is therefore:

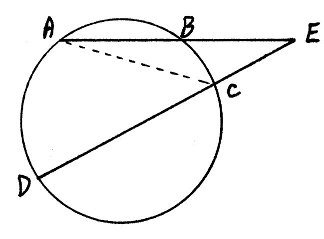


$$C = \tfrac{1}{2}(A - B)$$



Standard Proof: One secant crosses a circle at points A and B, and another secant crosses the circle at points C and D, such that the two secants intersect each other outside the circle, and the arcs BC and AD lie between the two secants. Draw chord AC. $\angle ACD = \angle BAC + \angle E$. ½arcAD = ½arcBC + $\angle E$.

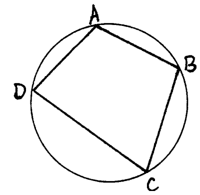
$$\therefore \angle E = \tfrac{1}{2}(\text{arcAD} - \text{arcBC})$$



Inscribed Quadrilateral Theorem.

"The opposite angles of a quadrilateral inscribed inside a circle are supplementary."

Proof:

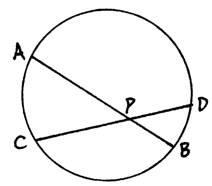


1. Draw quadrilateral ABCD inside a circle.
2. $\angle A = \frac{1}{2}$arcBCD; $\angle C = \frac{1}{2}$arcBAD
3. arcBCD + arcBAD = 360° (The whole circle!)
4. $\angle A + \angle C = \frac{1}{2}$(arcBCD + arcBAD) = ½(360°) = 180°
5. $\therefore \angle A$ and $\angle C$ are supplementary.

Chord Segment Theorem

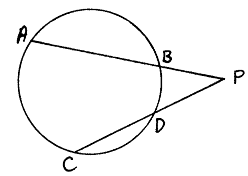
"When two chords intersect inside a circle, the product of the segments of one chord equals the product of the segments of the other chord."

Proof:



1. Given: circle with chords AB, CD intersecting at P.
2. Draw chords AD and BC.
3. $\angle A \cong \angle C$ and $\angle D \cong \angle B$
4. $\Delta APD \sim \Delta CPB$
5. AP:PD = CP:PB
6. $\therefore$ AP•PB = CP•PD

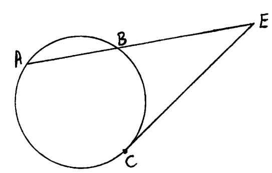
Secant Segment Theorem



"When two secant segments[1] are drawn from a point outside the circle, the product of one secant segment and its external portion equals the product of the other secant segment and its external portion."

Once again: AP•PB = CP•PD

Secant-Tangent Theorem

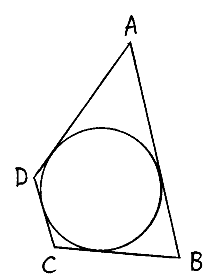


"When a tangent segment and a secant segment are drawn from a point outside a circle, the product of the secant segment and its external portion is equal to the square of the tangent segment."

Circumscribed Quadrilateral Theorem.

"The sum of the lengths of opposite pairs of sides of a circumscribed quadrilateral are equal."

Proof.



1. Draw quadrilateral ABCD, circumscribed (all four sides are tangent) to a given circle.
2. Color the two line segments (with the same color) from point A, along the side of the quadrilateral, to the two tangent points. Use different colors to color each pair of tangent line segments from the other three vertices.
3. Each pair of tangent line segments have equal lengths. (Intersecting tangent theorem.)
4. Any two opposite sides of the quadrilateral include one segment each of the four differently colored line segments.
5. $\therefore$The sum of the lengths of opposite pairs of sides are equal.

Required reading

In order to gain a deeper understanding of this *Circle Geometry* unit, the teacher should read the commentary found in the *Teacher's Edition of the 10th Grade Workbook* and also work through the *Circle Geometry* problem sets found in the workbook.

[1] A *secant segment* is a chord that has been extended in one direction to a point outside the circle.