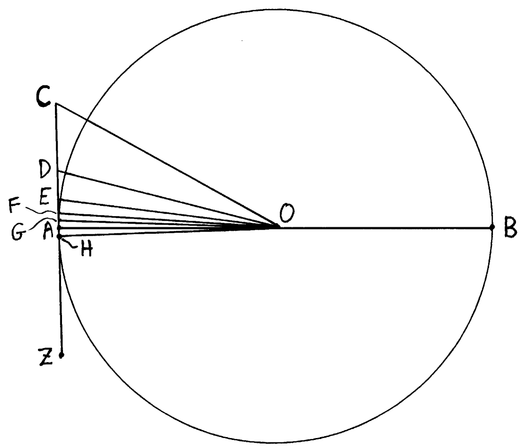
Archimedes' Calculation of π – as Archimedes did it[1]

Circumscribed Polygons



<u>Getting Started</u>

- With the above drawing, AC, AD, AE, AF, and AG are the <u>half-sides</u> of the circumscribed 6-gon, 12-gon, 24-gon, 48-gon, 96-gon, respectively.
- AB is a diameter of the circle, and O is the center. OA is perpendicular to CZ.
- Our goal is to find the ratio of OA:AG. We can then find the ratio of the perimeter of the 96-gon to the diameter of the circle.

<u>Calculating the 6-gon (hexagon)</u>

- Because ΔAOC is a 30°-60°-90° triangle, we know that $OA:AC = \sqrt{3}:1$ and $OC:AC = 2:1$
- Archimedes approximates $\sqrt{3}$ as $\frac{265}{153}$, which is too small by 0.0014%. We can then say:

OA:AC > 265:153 and **OC:AC = 306:153**

<u>Calculating the 12-gon</u>

- With ΔOAC, $\angle AOC$ is bisected by OD.
- OC:OA = CD:AD [Δ $\angle$ Bisector Th.; *Elem*. VI-3]
- (OC+OA):OA = (CD+AD):AD [*Elements* V-18]
- (OC+OA):OA = AC:AD
- (OC+OA):AC = OA:AD [*Elements* V-16]
- OA:AD = (OC+OA):AC
- By using the ratios from the 6-gon, we get: OA:AD > (306+265):153

  OA:AD > 571:153

  (The 12-gon is continued on next column…)

- $OD^2 = OA^2 + AD^2$ [Pythagorean Theorem]
- $OD^2 : AD^2 = (OA^2 + AD^2) : AD^2$
- $OD^2 : AD^2 > (571^2 + 153^2) : 153^2$
- $OD^2 : AD^2 > 349450 : 23409$

  OD:AD > $591\frac{1}{8}$: 153

<u>Calculating the 24-gon</u>

- We repeat the same process, but leave out the smaller steps this time.
- With ΔOAD, $\angle AOD$ is bisected by OE.
- OD:OA = DE:AE [Δ $\angle$ Bisector Th.; *Elem*. VI-3]
- OA:AE = (OD+OA):AD > ($591\frac{1}{8}$ + 571) : 153

  OA:AE > $1162\frac{1}{8}$: 153

- $OE^2 : AE^2 = (OA^2 + AE^2) : AE^2$
  $OE^2 : AE^2 > ({1162^1/_8}^2 + 153^2) : 153^2$
  $OE^2 : AE^2 > 1350534\frac{33}{64} : 23409$

  OE:AE > $1172\frac{1}{8}$: 153

<u>Calculating the 48-gon</u> (with even fewer steps!)

- OA:AF = (OE+OA):AE > ($1172\frac{1}{8}$ + $1162\frac{1}{8}$) :

  OA:AF > 2334¼ : 153

- $OF^2 : AF^2 = (OA^2 + AF^2) : AF^2$, which leads to:

  OF:AF > 2339¼ : 153

<u>Calculating the 96-gon</u> (finally!)

- OA:AG = (OF+OA):AF > (2339¼ + 2334¼) : 153

  OA:AG > 4673½ : 153 Our Goal!!

<u>Calculating an Upper Bound for π</u>

- Let P = Perimeter of 96-gon, and
  Let D = Diameter of circle.
- From above: AG:OA < 153 : 4673½
- π:1 < P : D = 192•AG : 2•OA = 96•AG : OA
- π:1 < 96•153 : 4673½ = 14688:4673½ < $3\frac{1}{7}$: 1

$$\pi < 3\tfrac{1}{7}$$

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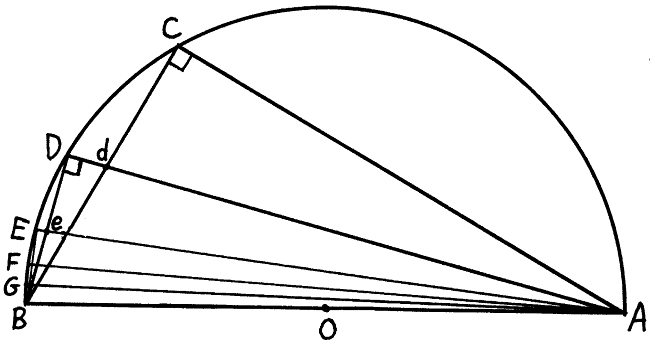
[1] Heath, T.L., *The Works of Archimedes*. New York: Dover Publications, 2002. pp 93-98.
This is the famous third theorem from Archimedes' work titled, *The Measurement of a Circle*.

Archimedes' Calculation of π – as Archimedes did it

Inscribed Polygons



Getting Started

- With the above drawing, the sides of the inscribed polygons, where the number of sides is 6, 12, 24, 48, 96, are given by BC, BD, BE, BF, BG, respectively.
- AB is a diameter of the circle, and O is the center.
- Our primary goal is to find the ratio of AB:BG. We can then find the ratio of the perimeter of the 96-gon to the diameter of the circle.

Calculating the 6-gon (hexagon)

- Because ΔABC is a 30°-60°-90° triangle, we know that $AC:BC=\sqrt{3}:1$ and $AB:BC=2:1$
- Archimedes approximates $\sqrt{3}$ as $\frac{1351}{780}$, which is too large by 0.000027%. We can then say:

AC:BC < 1351:780 and **AB:BC = 1560:780**

Calculating the 12-gon

- With ΔABC, ∠BAC is bisected by AD.
- AC:Cd = AB:Bd [Δ ∠ Bisector Th.; *Elem*. VI-3]
- AC:Cd = AB:Bd = (AC+AB):(Cd+Bd) [*Elements* V-12]
- AC:Cd = AB:Bd = (AC+AB):BC
- AC:Cd = AD:BD [because ΔACd ~ ΔADB]
- AD:BD = (AB+AC):BC [from above steps]
- By using the ratios from the 6-gon, we get: AD:BD < (1560 + 1351) : 780

AD:BD < 2911 : 780

- $AB^2 = AD^2 + BD^2$ [Pythagorean Theorem]
- $AB^2 : BD^2 = (AD^2 + BD^2) : BD^2$
  Using the above ratio, we get:
  $AB^2 : BD^2 < (2911^2 + 780^2) : 780^2$
  $AB^2 : BD^2 < 9082321 : 608400$

AB:BD < 3013¾ : 780

Calculating the 24-gon

- We repeat the same process, but leave out the smaller steps this time.
- With ΔABD, ∠BAD is bisected by AE.
- AD:De = AB:Be [Δ ∠ Bisector Th.; *Elem*. VI-3]
- AE:BE = (AB+AD):BD < (3013¾ + 2911) : 780
  AE:BE < 5924¾ : 780, which reduces to:

AE:BE < 1823 : 240

- $AB^2 : BE^2 = (AE^2 + BE^2) : BE^2$
  $AB^2 : BE^2 < (1823^2 + 240^2) : 240^2$
  $AB^2 : BE^2 < 3380929 : 57600$

AB : BE < 1838 $\frac{9}{11}$: 240

Calculating the 48-gon (with even fewer steps!)

- With ΔABE, ∠BAE is bisected by AF.
- AF:BF = (AB+AE):BE < (1838 $\frac{9}{11}$ + 1823) : 240
  AF:BF < 3661 $\frac{9}{11}$: 240, which reduces to:

AF:BF < 1007 : 66

- $AB^2 : BF^2 < (1007^2 + 66^2) : 66^2$
  $AB^2 : BF^2 < 1018405 : 4356$

AB : BF < 1009 $\frac{1}{6}$: 66

Calculating the 96-gon (finally!)

- With ΔABF, ∠BAF is bisected by AG.
- AG:BG = (AB+AF):BF < 2016 $\frac{1}{6}$: 66

AG:BG < 2016 $\frac{1}{6}$: 66

- $AB^2 : BG^2 < (2016\,{}^1/_6{}^2 + 66^2) : 66^2$
  $AB^2 : BG^2 < 4069284\frac{1}{36} : 4356$

AB : BG < 2017¼ : 66 Our goal!!

Calculating a Lower Bound for π

- Let P = Perimeter of 96-gon, and
  Let D = Diameter of circle.
- From above: BG : AB > 66 : 2017¼
- π:1 > P : D = 96•BG : AB
- π:1 > 96•66 : 2017¼ = 6336 : 2017¼ > $3\frac{10}{71}$: 1

$$\pi > 3\frac{10}{71}$$