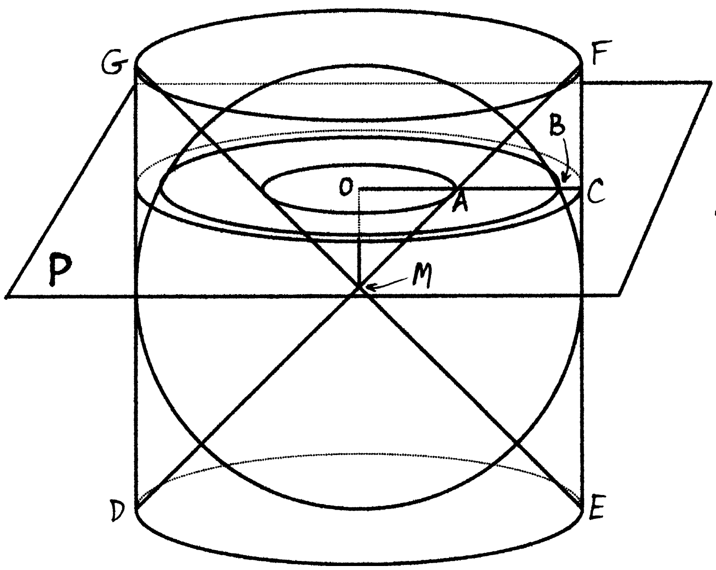
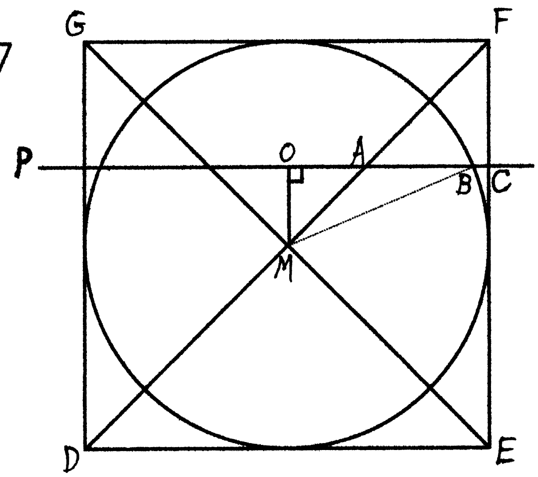
# Archimedes' Proof of the Formula for the Volume of a Sphere

View looking slightly downward




A vertical cross section through center, <u>or</u> a frontal view with eyes placed along plane P




## Proof:

1. Each figure shows the same cylinder, which has identical diameter and height. Inside the cylinder, sits a sphere with the same diameter, and also a double cone, again with the same height and diameter. The sphere and cone interpenetrate one another.
   (Note: Volume of double cone = volume of regular cone)
2. Plane P cuts horizontally through the cylinder creating three circles, which are the cross sections of the cylinder, sphere, and cone. Depending on where plane P cuts through the cylinder (in other words, how close plane P is to M), the circle formed by the cross section of the sphere (on plane P) may be larger or smaller than the cross section formed by the cone. The radii of the three circles are:

   **$r_1$ = OA** (radius of the circle formed by cross section of the cone)
   **$r_2$ = OB** (radius of the circle formed by cross section of the sphere)
   **$r_3$ = OC** (radius of the whole cylinder, or of its cross section)
3. Radius of the *whole* sphere = the radius of the cylinder. Therefore:
   OC = **MB = $r_3$**
   $\angle OMA = \angle OAM = 45°$; OA = **OM = $r_1$**
   We now have a right triangle: $\Delta OMB$.
   $MB^2 = OB^2 + OM^2 \rightarrow r_3^2 = r_2^2 + r_1^2 \rightarrow \pi r_3^2 = \pi r_2^2 + \pi r_1^2$
   $\therefore$ The sum of the areas of the two smaller circles is equal to the area of the largest circle.
4. Consider a second plane $P_1$, parallel to P, and separated from P by a very small distance $\Delta x$. Three cross-sectional disks are now formed between the two planes - one each for the cone, sphere, and the cylinder. We can say that the sum of the volumes of the disks from the cone and the sphere is equal to the volume of the disks from the cylinder.
5. Consider that the cone, sphere, and cylinder are all a stack of such thin disks. Then the volume of the whole cone plus the volume of the whole sphere is equal to the volume of the whole cylinder. And since the volume of the cone is ⅓ of the cylinder, the volume of the sphere is then ⅔ of the cylinder.

   Archimedes expresses this as a ratio of the volumes: **Cone:Sphere:Cylinder = 1:2:3**