—— *Al-Khwarizmi's Algebra* (9th Grade) ——

Note: Muhammad ibn Musa al-Khwarizmi is known as the "father of algebra". His work, *Hisab al-jabr wal-muqabala*, written in 825AD, is considered to be the first book on algebra. These sections of al-Khwarizmi's book are found in our *9th Grade Workbook*; it focuses on his derivation of a version of the quadratic formula. I have reworded much of the text from Frederic Rosen's translation of *The Algebra of Mohammed Ben Musa* (pp 5-21).

On Calculating by Completion and Reduction

INTRODUCTION

When I considered what people generally want in calculating, I found that it is always a number. I also observed that every number is composed of units, and that any number may be divided into units.

Furthermore, I observed that the numbers which are required when calculating by completion and reduction are of three kinds, namely: roots, squares, and simple numbers[1]. Of these, a root is any quantity which is to be multiplied by a number greater than unity, or by a fraction less than unity. A square is that which results from the multiplication of the root by itself. A simple number [henceforth called only "number"] is any number which may be produced without any reference to a root or a square.

Of these three forms, then, two may be equal to each other, for example: squares equal to roots, squares equal to numbers, and roots equal to numbers[2].

Section I. CONCERNING SQUARES EQUAL TO ROOTS

The following is an example of squares equal to roots: "A square is equal to five roots". The root of the square then is five, and twenty-five forms its square, which is indeed equal to five times its root.

Another example: "One-third of a square equals four roots." Then the whole square is equal to 12 roots. So the square is 144, and its root is 12. Another such example: "Five squares equal ten roots." Therefore one square equals two roots. So the root of the square is two, and four represents the square.

In this manner, that which involves more than one square, or is less than one square, is reduced to one square. Likewise, the same is done with the roots; that is to say, they are reduced in the same proportion as the squares.

Section II. CONCERNING SQUARES EQUAL TO NUMBERS

The following is an example of squares equal to numbers: "A square is equal to nine." Then nine is the square and three is the root. Another example: "Five squares equal 80." Therefore one square is equal to one-fifth of 80, which is 16. Or, to take another example: "Half of a square equals 18." Then the whole square equals 36, and its root is six.

Thus any multiple of a square can be reduced to one square. If there is only a fractional part of a square, you multiply it in order to create a whole square. Whatever you do, you must do the same with the number.

Section III. CONCERNING ROOTS EQUAL TO NUMBERS

The following is an example of roots equal to numbers: "A root is equal to three." Then the root is three and the square is nine. Another example: "Four roots equal 20." Therefore one root is five, and the square is 25. Still another example: "Half a root is equal to ten." Then the whole root is 20 and the square is 400.

[In addition to the three above cases] I have found that these same three elements can produce three compound cases, which are:

Squares and roots equal to numbers,

Squares and numbers equal to roots, and

Roots and numbers equal to squares.

[These three above cases are variations of a quadratic equation. Sections IV, V, and VI give methods for solving each of these cases. Only section IV, which deals with the first case, is included in this source book and in our *9th Grade Workbook*.]

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[1] The term "roots" stands for multiples of the unknown, our x; the term "squares" stands for multiples of our x^2; "numbers" are constants.

[2] In our modern notation, this is $x^2 = bx$, $x^2 = c$, $x = c$.

Section IV. CONCERNING SQUARES AND ROOTS EQUAL TO NUMBERS

The following is an example of squares and roots equal to numbers: "A square and ten roots are equal to 39." The question here is: "What must the square be such that when it is combined with ten of its own roots, it will amount to a total of 39?" To solve this, you take half the number of roots, which in this case gives us five. Then you multiply this by itself to get 25, and then add that result to 39, which gives us 64. Now take the root of this, which is eight, and subtract from it half the number of roots, resulting in three. This is the root of the square which you sought; the square itself is then nine.

This method is the same when you are given a number of squares. You simply reduce them to a single square, and in the same proportion you reduce the roots and simple numbers that are connected with them.

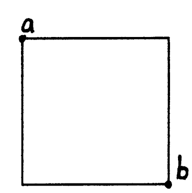
For example: "Two squares and ten roots equal 48." The question therefore is: "What must the amount of the two squares be such that when they are summed up and then combined with ten times their root, the result will be a total of 48?" First of all it is necessary that the two squares be reduced to one. So we take half of everything mentioned in the statement. It is the same now as if the original question had been: "A square and five roots equal 24", which means: "What must the amount of a square be such that when it is combined with five times its root, the result will be a total of 24?" To solve this, we halve the number of roots, which gives us 2½, and multiply that by itself, giving 6¼. To this we add 24, which yields a sum of 30¼, and then take the root of this, which is 5½. Subtracting half the number of roots, which is 2½, from this makes a remainder of three. This is the root of the square, and the square itself is nine.

[In section V Al-Khwarizmi gives a solution for solving *Squares and Numbers Equal to Roots* ($10x = x^2 + 21$). Section VI gives a solution for solving *Roots and Numbers Equal to Squares* ($3x + 4 = x^2$). We are skipping over these two sections. We will now pick up with the last paragraph of section VI, which reads as follows:]

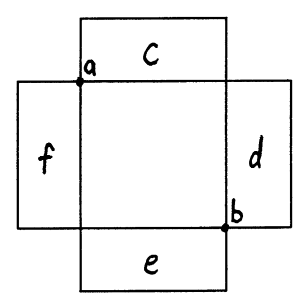
I have now explained the six types of equations, which I first mentioned at the beginning of this book. I have taught how the first three must be solved; with these, it was not required that the roots be halved. And I showed how, with the other three, halving the roots is necessary. I now think it is necessary to explain the reason for halving.

Section VII. A DEMONSTRATION OF THE CASE "*A Square and Ten Roots Equal 39.*"

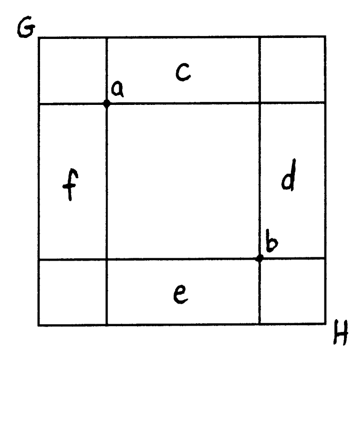
First, we construct a square ab of unknown sides. This square represents the square which, together with its root, you wish to find. Any side of this square represents one of the roots that we wish to know. We will now take one-fourth of the number of roots, namely one-fourth of ten, to get 2½. Combining this with the side of the square gives us four new rectangles (c, d, e, f), which we will place onto the sides of the square [as shown in the middle drawing].



We now have a new, larger square except that small square pieces are missing from its four corners. These four corners each have an area of 2½ times 2½. When we add these four corners to our figure [as shown in the lower drawing], we have increased the area by four times the square of 2½, which is 25.



From the original statement we know that the square ab combined with the four rectangles, which together represent ten roots, must be equal to a total of 39. To this we add 25 (the area of the four small corners) to get a total of 64, which is the area of the great square GH. One side of this great square must then be eight. If we subtract twice a fourth of ten, which is five, from this eight then we get three – the root of the square which we sought.

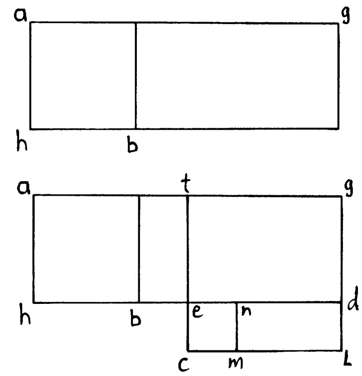


It must be observed that here we have taken one-fourth the number of roots, multiplied that result by itself, and then multiplied that by four, which is the equivalent of taking half the number of roots and then multiplying that by itself [which is what was done in section IV].

Section VIII. A DEMONSTRATION OF THE CASE "*Ten Roots Equals a Square and 21.*"

[This is our rewording of Section VIII.[1] This section *is not* found in our *9th Grade Workbook*.]

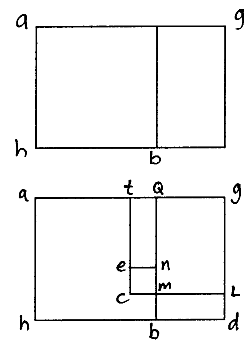
This is the type of quadratic equation that yields two (positive) solutions. The two solutions emerge from whether we draw the rectangle having a height more or less than half of the base (assuming the base is greater than the height).



Solution #1 (The height of the rectangle is less than half its base.)

First, we draw square *ab* (which represents x^2) and then attach to it rectangle *bg* (which represents the 21 units) so that it is larger than the square *ab*. This now forms a larger rectangle *hg* (which represents 10x). At this moment, the diagram clearly represents the equation $10x = x^2 + 21$. Our goal is to determine x, the length of the side of the square, *ah*.

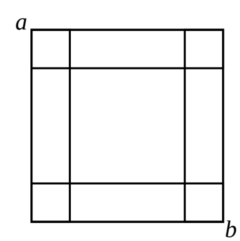
Al-Khwarizmi bisects *ag* at *t*, constructs square *cg*, and then constructs square *nc* (so that *en* = *ec*). The rectangle *tb* is congruent to rectangle *md*. This means that rectangle *bg* and gnomon *tenmlg* both have equal areas of 21. Since *ag* is 10, *tg* is 5, and the square *tg* has an area of 25. Now, the square *nc* has an area of 4 (the area of the square *cg* minus the area of the gnomom). So, *ec* must be 2, and *tc* minus *ec* then gives a length of **3** for both *te* and *ah*, which is the desired length of the side of the square *ab*.



Solution #2 (The height of the rectangle is greater than half its base.)

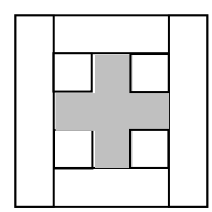
This time we begin our drawing by making rectangle *bg* smaller than the square *ab*. Our goal, once again, is to find the length of the side of the square, *ah*. Again, we bisect *ag* at *t*, construct square *cg*, and then construct square *nc* (so that *en* = *ec*). The rectangle *tn* is congruent to rectangle *md*. This means that rectangle *bg* and gnomon *tenmlg* both have equal areas of 21. Since *ag* is 10, *tg* is 5, and the square *tg* has an area of 25. Now, square *nc* has an area of 4 (the area of the square *cg* minus the area of the gnomom). So, *nm* must be 2, which is also the length of *mb*. Therefore, Q*m* (which is 5) plus *mb* gives a length of **7** for Q*b*, which is also the desired length of *ah*, the side of the square *ab*.

Section IX. A DEMONSTRATION OF THE CASE "*A Square Equals Three Roots and 4.*"



[Al-Khwarizmi uses a diagram similar to the last one in order to prove this last case. I, however, have come up with the following proof, which is more similar to his first proof given in section VII, which is the best known. This section *is not* found in our *9th Grade Workbook*]

We start with square *ab*. Our goal is to find the length of the side of this square. Since the rectangle ("three roots") is smaller than the square, we cut it parallel to its base into four thin strips and then put these four strips *inside* the square such that where they overlap forms four small squares at the corners of the larger square (as shown on the right). Given that the original rectangle had a width of 3, we know that the small squares each have an area of $(3/4)^2$, which is $9/16$. If we move all of these small squares toward the center (shown in the lower drawing) then the rectangle has now been cut into eight pieces that fit inside the original, large square in a way that the pieces no longer overlap. Since the original problem states that the area of the square is 4 greater than the area of the rectangle, the shaded region represents how much larger the original square is than the rectangle. The shaded region thus has an area of 4, and when combined with the four small corner squares (each with an area of $9/16$) we get a medium-sized square (the shaded area sits inside this square) with an area of 6¼. The side of this square is then the square root of 6¼, which is 2½. By looking at the diagram, we can see that the desired length of the side of the original square *ab* must be 2½ + ¾ + ¾, which is **4**.



[1] See a *History of Mathematics*, by Carl Boyer, pp231-232, John Wiley & Sons, 1991.